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The chiral ring of AdS_3/CFT_2 and the attractor mechanism

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Abstract: We study the moduli dependence of the chiral ring in $\mathcal{N} = (4,4)$ superconformal field theories, with special emphasis on those CFT's that are dual to type IIB string theory on $AdS_3 \times S^3 \times X_4$. The chiral primary operators are sections of vector bundles, whose connection describes the operator mixing under motion on the moduli space. This connection can be exactly computed using the constraints from $\mathcal{N} = (4,4)$ supersymmetry. Its curvature can be determined using the tt^* equations, for which we give a derivation in the physical theory which does not rely on the topological twisting. We show that for $\mathcal{N} = (4,4)$ theories the chiral ring is covariantly constant over the moduli space, a fact which can be seen as a non-renormalization theorem for the three-point functions of chiral primaries in AdS_3/CFT_2. From the spacetime point of view our analysis has the following applications. First, in the case of a D1/D5 black string, we can see the matching of the attractor flow in supergravity to RG-flow in the boundary field theory perturbed by irrelevant operators, to first order away from the fixed point. Second, under spectral flow the chiral primaries become the Ramond ground states of the CFT. These ground states represent the microstates of a small black hole in five dimensions consisting of a D1/D5 bound state. The connection that we compute can be considered as an example of Berry's phase for the internal microstates of a supersymmetric black hole.

Keywords: Supersymmetry and Duality, Gauge-gravity correspondence, AdS-CFT Correspondence, Conformal Field Models in String Theory

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1 Introduction

The $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence [1] is one of the best understood holographic dualities and has been very useful for the analysis of black holes in string theory. While it has been studied in great detail by now, most of the computations have been performed in special weakly-coupled limits. The $\mathrm{AdS}_3/\mathrm{CFT}_2$ is characterized by a parameter space $\mathcal{M}$ which corresponds to the expectation values of the scalar fields in the bulk, or equivalently to the position on the moduli space of the boundary CFT. There are special points on $\mathcal{M}$ where the boundary CFT is weakly coupled and others where the holographic dual string theory is in the perturbative regime. At a generic point on $\mathcal{M}$, none of the two descriptions is weakly coupled and it is difficult to make any explicit computations. Is there anything we can say about the theory in the interior of its moduli space?

In this paper, whenever we speak of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, we will have the duality between type IIB on $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{X}_4$ and suitable $\mathcal{N}=(4,4)$ superconformal field theories in mind. These CFT's are believed to be related to a sigma model whose target space is a deformation of the symmetric product X^N/S_N, where $X=T^4$ or $K3$. This is a hyperkähler space and such sigma models are indeed compatible with $\mathcal{N}=(4,4)$ sypersymmetry. It is a natural assumption that at all points on $\mathcal{M}$ the theory has a boundary description in terms of an $\mathcal{N}=(4,4)$ superconformal field theory, which may be strongly coupled. Such theories have a sector protected by supersymmetry, the chiral

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ring [2], which can be studied exactly even away from the weak-coupling limits. In this paper we analyze the moduli dependence of the chiral ring of $\mathcal{N} = (4,4)$ superconformal field theories, mainly motivated by its relevance for the boundary CFT that appears in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. Our analysis is exact everywhere on the moduli space, since we only assume that the $\mathcal{N} = (4,4)$ superconformal structure of the theory is preserved and that generically the number of chiral primaries does not jump as we move on $\mathcal{M}$. This allows us to make some exact statements about the theory in the regime of strong coupling and for finite N.

The chiral ring of a superconformal field theory depends on the moduli in two ways. First, the chiral primaries mix among themselves as we change the parameters of the theory. Technically this means that the chiral primary operators are sections of vector bundles over the moduli space, which can have nontrivial curvature. Second, the multiplication between the chiral primaries, expressed in terms of the structure constants C^i_{jk}, may also be moduli dependent. Supersymmetry imposes strong constraints on the structure of the chiral ring and the way it behaves under a change of the coupling constants. The case of $\mathcal{N} = (2,2)$ superconformal theories has been extensively studied and the supersymmetry constraints are expressed in terms of the tt^* equations

$$R_{i\bar{j}} \equiv [\nabla_i, \nabla_{\bar{j}}] \simeq -[C_i, \overline{C_j}] \tag{1.1}$$

which give the curvature of the bundles of chiral primaries in terms of the chiral ring coefficients. These equations were originally derived by Cecotti and Vafa using a method called topological anti-topological fusion [3, 4] which is based on the topological twisting of the superconformal theory. However as we show they can also be derived using ordinary conformal perturbation theory in the untwisted theory.[1]

The tt^* equations are also relevant for theories with $\mathcal{N} = (4,4)$ supersymmetry, such as the boundary theory in the class of $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondences we consider here, once we appropriately project to their $\mathcal{N} = (2,2)$ subalgebras. In $\mathcal{N} = (4,4)$ theories a simple observation leads to the following additional constraint

$$\nabla C^k_{ij} = 0 \tag{1.2}$$

where ∇ represents the covariant derivative along any marginal deformation. This is true for the following reason: in an $\mathcal{N} = (2,2)$ theory it is known that the chiral ring coefficients depend on the moduli holomorphically, so they are independent of anti-holomorphic deformations

$$\nabla_{\overline{m}} C^k_{ij} = 0 \tag{1.3}$$

while in general $\nabla_m C^k_{ij} \neq 0$. An $\mathcal{N} = (4,4)$ theory has many inequivalent $\mathcal{N} = (2,2)$ subalgebras. It can be shown that in an $\mathcal{N} = (4,4)$ theory any marginal deformation can be written as an anti-holomorphic deformation with respect to some $\mathcal{N} = (2,2)$ subalgebra, and then (1.2) follows from (1.3).

[1] The original derivation is more general since it also works for non-conformal $\mathcal{N} = (2,2)$ theories.

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The result (1.2) can be interpreted as a non-renormalization theorem for the 3-point functions of chiral primaries in AdS_3/ CFT_2, which explains the agreement of computations performed at different points on the moduli space [5, 6] and also [7, 8]. This is the analogue of the non-renormalization theorem [9, 10] for 3-point functions of chiral primaries in AdS_5/CFT_4 which explained the agreement of the weakly and strongly coupled computations [11]. Our arguments do not depend on taking a large N limit, so the 3-point functions of chiral primaries have to be (covariantly) constant even at finite N. It is easy to show that more generally *extremal* correlators of chiral primaries are also not renormalized as we change the moduli.

Combining the non-renormalization of the chiral ring coefficients with the tt^* equations we can derive a stronger statement. By acting with ∇ on both sides of (1.1) and using (1.2) we conclude that the curvature of the bundle of chiral primaries is covariantly constant

$$\nabla R_{i\bar{j}} = 0 \tag{1.4}$$

We also know [12, 13] that for $\mathcal{N} = (4,4)$ theories the moduli space is locally a symmetric space of the form

$$\frac{\mathrm{SO}(4,n)}{\mathrm{SO}(4) \times \mathrm{SO}(n)} \tag{1.5}$$

for some n. Bundles with covariantly constant curvature over symmetric spaces are called *homogeneous bundles* and their geometry is completely determined in terms of some basic group-theoretic data. In some $\mathcal{N} = (4,4)$ theories, such those that arise in the AdS_3/CFT_2 correspondence, if we know the number of chiral primaries of a given conformal dimension, it is rather straightforward to fit them into homogeneous bundles. Then the exact connection and curvature on these bundles is determined without any further input from the dynamics of the CFT. In this sense we can compute the exact mixing of chiral primary operators as we move on the moduli space, even at strong coupling.

An application of our analysis from the spacetime point of view is that it realizes a connection between the attractor flow in supergravity and RG-flow in the boundary field theory, in a certain toy-model, as we now explain. Extremal black holes in supergravity exhibit a remarkable phenomenon, called the attractor mechanism [14]. The values of many of the scalar fields near the horizon of the black hole are fixed by its electric and magnetic charges and completely independent of their values at spatial infinity. The same black holes can be described by appropriate bound states of D-branes. The worldvolume theory of these branes is an open string theory, which flows to a conformal field theory at low energies. This raises a natural question, namely what is the meaning of the attractor flow in the D-brane picture of the black hole?

As is well known, the AdS throat of the supergravity solution is holographically dual to the conformal IR fixed point of the effective field theory describing the excitations on the D-branes that create the black hole. The AdS/CFT correspondence is derived by taking the low energy limit which on the supergravity side is equivalent to keeping only the near horizon AdS geometry. In that region of the supergravity solution the moduli have already reached their attractor values. As a result the attractor mechanism is not visible in the usual AdS/CFT correspondence.

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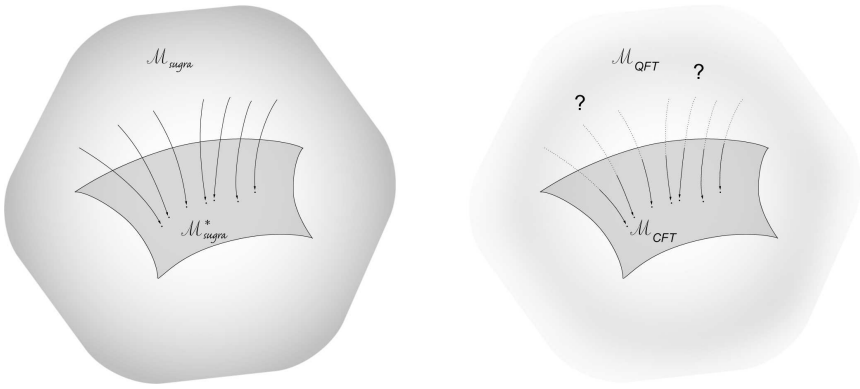

Figure 1. Attractor flow in supergravity (left) and RG-flow on the worldvolume of the branes (right).

Clearly, to see the attractor flow we have to move outside the AdS throat towards the asymptotically flat region. This requires an extension of AdS/CFT beyond the strict $\alpha' \to 0$ limit, where it turns into a duality between closed string theory and open string theory. In the open string language the system is described by a stack of D-branes in flat space, and on the other hand, in the large N limit, we can consider the closed string description where we replace the D-branes by a curved closed string background. On the boundary side going outside the AdS throat is described by deforming the CFT by irrelevant operators. From this perspective we expect to see the attractor flow as RG-flow on the worldvolume theory of the branes towards the IR fixed point. See also [15] for related discussions.

It is not easy to make this relation precise, since going outside the AdS throat means that there is no honest decoupling between open and closed string modes. In particular, since the open strings living on the branes are not decoupled from the bulk closed string modes it is not clear what we mean by the "boundary theory". However as we approach the IR fixed point, this coupling should become less and less important. In this sense we expect that at least near the fixed point it should be possible to describe the theory on the branes in terms of an effective field theory flowing to a CFT in the IR. In view of these conceptual difficulties we will only consider the first order perturbation away from the conformal point towards the UV, which should correspond to the final stages of the attractor flow. More precisely, as shown in figure 1, let us call $\mathcal{M}_{\rm sugra}$ the moduli space of supergravity and $\mathcal{M}^*_{\rm sugra} \subset \mathcal{M}_{\rm sugra}$ the attractor submanifold for given charges.[2] On the boundary side we have a family of effective quantum field theories characterized by a moduli space $\mathcal{M}_{\rm QFT}$ which flow in the IR to a family of conformal field theories with moduli space $\mathcal{M}_{\rm CFT} \subset \mathcal{M}_{\rm QFT}$. According to the AdS/CFT correspondence the moduli spaces $\mathcal{M}^*_{\rm sugra}$ and $\mathcal{M}_{\rm CFT}$ should be identical.[3] Moreover, matching the final stages of the attractor flow

[2]We would like to remind that even in Calabi-Yau compactifications of type II, while the attractor equations fix the vector multiplets to discrete points, the hypermultiplets are unfixed, so also in this case there is a continuous family of attractor points parametrized by the hypermultiplet moduli space.

[3]This has been demonstrated in some examples of $\mathrm{AdS}_3/\mathrm{CFT}_2$ [16]. We would expect the same for other cases, such as the MSW CFT [17]. In the case of 4d black holes and $\mathrm{AdS}_2/\mathrm{CFT}_1$ the equivalent statement would be that the "moduli space" of the superconformal quantum mechanics must be the same as the

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to RG-flow means that the normal bundle of $\mathcal{M}^*_{\rm sugra}$ inside $\mathcal{M}_{\rm sugra}$ should have the same structure as that of $\mathcal{M}_{\rm CFT}$ inside $\mathcal{M}_{\rm QFT}$.[4] In particular, this means that the dimensionality of the bundles should agree, in other words - we should have the same number of irrelevant operators as the number of scalar moduli fixed by the attractor mechanism, and in addition the connection on the two bundles should be the same.

This picture is easy to check in the simple case of the attractor flow near an extremal black string in six dimensions. In this case the boundary CFT is the one appearing in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. As we will see the irrelevant operators which preserve supersymmetry are descendants of certain fields in the chiral ring. Hence, their number can be counted and moreover the connection and geometry of their bundle can be exactly computed using our general analysis. The result that we find on the CFT side agrees with the predictions from the attractor flow in supergravity.

Finally let us mention another interpretation of the geometry of the chiral ring that we study in this paper. Spectral flow relates the chiral primaries of the CFT to Ramond ground states. In the D1/D5 CFT, the Ramond ground states have the following interpretation. We consider IIB compactified on $K3 \times S^1$ and a bound state of D1/D5 branes wrapped on the internal manifold. This looks like a small supersymmetric black hole in five dimensions. The Ramond ground states of the CFT represent the internal microstates of the black hole. If we adiabatically change the moduli of the compactification the microstates will mix among themselves, as is well known from the non-abelian generalization of Berry's phase for quantum mechanical systems with degenerate microstates. The connection for the chiral primaries is related to the connection of the Ramond ground states over the moduli space, in other words it yields a geometric phase for the internal microstates of the black hole.

In the first half of the paper we review background material. In section 2 we review some basic facts about the chiral ring in superconformal field theories. In section 3 we discuss the deformation of conformal field theories by marginal operators and the associated connections for the bundle of operators over the moduli space. In section 4 we review basic results for the connection of the bundle of chiral primaries for the case of $\mathcal{N} = (2,2)$ theories and show how the tt^* equations follow from conformal perturbation theory. In section 5 we introduce the $\mathcal{N} = (4,4)$ algebra and discuss its basic properties. In section 6 we show that the 3-point functions in $\mathcal{N} = (4,4)$ theories are covariantly constant and we compute the curvature for the bundle of chiral primaries. In section 7 we present the relevance of our computation for the connection between the attractor flow and RG-flow. In section 8 we discuss how the connection for chiral primaries is related to Berry's phase for black hole microstates. In section 9 we summarize our results and discuss some possibilities for future research.

hypermultiplet moduli space. It would be interesting to give a more precise meaning to this statement.

[4] In general the geometry of $\mathcal{M}^*_{\rm sugra}$ will receive corrections beyond supergravity, which have to be taken into account in order to achieve a precise matching with the CFT moduli space. This does not happen in the D1/D5 system due to the extended supersymmetry.

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2 AdS_3/CFT_2 and its chiral ring

2.1 Generalities

We can derive the AdS_3/CFT_2 correspondence with 16 supercharges by starting with IIB string theory compactified on X, where $X = T^4$ or $K3$ and considering a BPS black string in six dimensions, consisting of a bound state of D1 strings and D5 branes wrapped on X. By taking a low energy decoupling limit of this system we find the duality between IIB on

$$AdS_3 \times S^3 \times X \tag{2.1}$$

and a two dimensional CFT with $\mathcal{N} = (4,4)$ supersymmetry and $\mathrm{SU}(2)^R_{\mathrm{left}} \times \mathrm{SU}(2)^R_{\mathrm{right}}$ as its current algebra. Excluding the center of mass degrees of freedom, the level k and central charge c are given by

$$k = Q_1 Q_5 \qquad c = 6 Q_1 Q_5. \tag{2.2}$$

This CFT can be understood as a supersymmetric sigma model whose target space is a resolution of the symmetric product X^N/S_N with $N = Q_1 Q_5$, which is moduli space of instantons of degree Q_1 of a $\mathrm{U}(Q_5)$ gauge theory living on X.

The AdS_3/CFT_2 correspondence is characterized by the integer c and by a set of continuous parameters determined by the background values of the moduli fields of IIB. In other words, the correspondence has a moduli space[5] $\mathcal{M}$. This moduli space is visible on the boundary side as the moduli space of the conformal field theory $\mathcal{M}_{\mathrm{CFT}}$ and on the bulk side as the moduli space $\mathcal{M}^*_{\mathrm{sugra}}$ of possible values of the scalar fields near the horizon of the black string in 6d. The local structure of the moduli space is exactly computable from both sides of the duality [16] and it is of the form

$$\mathcal{M} \simeq \frac{\mathrm{SO}(4,n)}{\mathrm{SO}(4) \times \mathrm{SO}(n)} \tag{2.3}$$

where $n = 5$ for $X = T^4$ and $n = 21$ for $X = K3$.

Notice that this is a local statement. The global structure of $\mathcal{M}$ is more complicated [16, 18] ,[19] and there are points where the CFT is singular. In this paper we will only consider local properties and ignore all subtleties related to the global structure of the moduli space and possible monodromies around singularities.

There are points of $\mathcal{M}$ where the boundary CFT is weakly coupled. It is believed that there is a point where the CFT can be described as a symmetric orbifold CFT [20, 21], which is analogous to the $\lambda \to 0$ limit in AdS_5/CFT_4. There are other points of $\mathcal{M}$ where the bulk side of the correspondence is at weak coupling and where it is possible to perform computations in weakly coupled string theory or supergravity.

Once we consider the theory away from these special limits, at a generic point in the interior of the moduli space $\mathcal{M}$, it is hard to compute anything exactly since both the bulk and the boundary sides have coupling constants of order unity. However, as long as we stay

[5] Similarly the AdS_5/CFT_4 has the discrete parameter N and the continuous parameter $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2_{\mathrm{YM}}}$.

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away from singularities, it is reasonable to assume that at all points of $\mathcal{M}$ the theory has a boundary description in terms of a 2-dimensional conformal field theory with $\mathcal{N} = (4,4)$ supersymmetry and central charge given by (2.2).

In any conformal field theory with extended supersymmetry, and in particular in the boundary theory of $\mathrm{AdS}_3/\mathrm{CFT}_2$, there is a protected sector consisting of chiral primary operators. These operators form a ring under multiplication and their 3-point functions characterize the structure of the ring [2]. The chiral primaries in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence have been identified in the weak-coupling limits of $\mathcal{M}$. The counting of their degeneracies in the orbifold CFT limit is in agreement with their counting from supergravity [22, 23]. More surprisingly, their 3-point functions, that is the structure of the chiral ring, is the same at different points of the moduli space [5–8].

Our goal is to compute the moduli dependence of the chiral ring at a generic point of $\mathcal{M}$, where no weakly coupled description of the theory is available. This is possible due to the extended supersymmetry. As we will see, the chiral ring is covariantly constant over $\mathcal{M}$. In particular, we will understand the non-renormalization theorem for the 3-point functions of chiral primaries in $\mathrm{AdS}_3/\mathrm{CFT}_2$. In the rest of this section we will review some background material.

2.2 Chiral primaries and the chiral ring

We start with a quick review of the chiral ring of 2-dimensional superconformal field theories [2]. Ultimately we are interested in $\mathcal{N} = (4,4)$ theories, but for simplicity of notation in this section we will only consider the left-moving part of an $\mathcal{N} = (2,2)$ SCFT.

The left-moving currents are the energy momentum tensor $T(z)$, two supercurrents $G^{\pm}(z)$ and the U(1) R-current $J(z)$. The superscript index of the supercurrents denotes their R-charge which is ± 1. An operator ϕ is called superconformal primary if it satisfies the condition

$$L_n|\phi\rangle = J_n|\phi\rangle = G^+_{n-\frac{1}{2}}|\phi\rangle = G^-_{n-\frac{1}{2}}|\phi\rangle = 0, \qquad n > 0. \tag{2.4}$$

If in addition it satisfies

$$G^+_{-1/2}|\phi\rangle = 0 \tag{2.5}$$

then it is called *chiral primary*. Using the $\mathcal{N} = 2$ algebra we can show that for such operators we have

$$(2L_0 - J_0)|\phi\rangle = 0 \tag{2.6}$$

and therefore the conformal dimension h and the R-charge q are related as $h = q/2$. Conversely, in a unitary CFT we can show that a primary field satisfying (2.6) will be chiral. Similarly we define antichiral primary fields $\overline{\phi}$ which satisfy

$$G^-_{-1/2}|\overline{\phi}\rangle = 0 \qquad \Leftrightarrow \qquad (2L_0 + J_0)|\overline{\phi}\rangle = 0. \tag{2.7}$$

Their dimension and R-charge are related by $h = -q/2$. Obviously if a field ϕ is chiral, then $\phi^\dagger$ is antichiral.

A remarkable property of chiral primary operators is that they form a ring. The OPE of two chiral primaries is nonsingular as can be demonstrated by U(1) charge conservation

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and unitarity and has the form

$$\phi_i(z)\phi_j(w) = C_{ij}^k \phi_k(w) + \dots \tag{2.8}$$

where the operator ϕ_k is also chiral primary of charge $q_k = q_i + q_j$. The constants C_{ij}^k are the structure constants of the ring.

We define the two point function of chiral primaries on the sphere which plays the role of Zamolodchikov's metric

$$\langle \phi_i(0)\overline{\phi_j}(\infty)\rangle = g_{i\bar{j}}, \tag{2.9}$$

and can be nonzero only if the fields have opposite R-charges. We also have the 3-point functions on the sphere

$$\langle \phi_i(0)\phi_j(1)\overline{\phi_k}(\infty)\rangle = C_{ij\bar{k}} \tag{2.10}$$

where again from charge conservation it must be of the form chiral-chiral-antichiral.

Using the OPE of the chiral ring we find the following relation between the chiral ring coefficients and the 3-point functions

$$C_{ij\bar{k}} = C_{ij}^l g_{l\bar{k}}. \tag{2.11}$$

Our discussion up to this point has been about the left-moving sector of an $\mathcal{N} = 2$ theory. When we consider the full $\mathcal{N} = (2,2)$ theory we can have fields which are chiral on both sides, antichiral on both, or chiral - antichiral, and we will have four corresponding rings $(cc), (aa), (ca), (ac)$ which are pairwise complex conjugate.

As we will explain in more detail later, for $\mathcal{N} = (4,4)$ theories we can use the enhanced R-symmetry $\mathrm{SU}(2)^R_{\text{left}} \times \mathrm{SU}(2)^R_{\text{right}}$ to rotate a chiral field into an antichiral one. This implies that all four rings are equivalent, so essentially there is only one ring in an $\mathcal{N} = (4,4)$ theory.

2.3 Moduli dependence

So far we have considered the chiral primaries and their OPEs in a given SCFT. Usually superconformal field theories come in families, parametrized by a moduli space $\mathcal{M}_{\text{CFT}}$. Motion along $\mathcal{M}_{\text{CFT}}$ is generated by marginal operators. We will consider perturbations by operators which preserve the $\mathcal{N} = (2,2)$, or $\mathcal{N} = (4,4)$, structure and we will stay away from any singularities on the moduli space, so we will assume that $\mathcal{M}_{\text{CFT}}$ is a smooth manifold of fixed dimension, at least locally.

While the dynamics of the CFT depends on the position on $\mathcal{M}_{\text{CFT}}$, certain properties of the chiral ring are protected. For example the number of chiral primaries of given dimension is generally constant on $\mathcal{M}_{\text{CFT}}$. It is possible for chiral primaries to pair up into long multiplets and leave the BPS spectrum, but this will happen at special points or submanifolds of the moduli space. We will restrict our analysis to regions of $\mathcal{M}_{\text{CFT}}$ where this does not happen. In $\mathrm{AdS}_3/\mathrm{CFT}_2$ this assumption is justified by the agreement of the counting of chiral primaries in the symmetric orbifold and the supergravity limits.

In a general $\mathcal{N} = (2,2)$ SCFT the structure constants C_{ij}^k and the 2- and 3-point functions $g_{i\bar{j}}$, $C_{ij\bar{k}}$ are usually nontrivial functions on $\mathcal{M}_{\text{CFT}}$. The agreement of 3-point functions of chiral primaries in $\mathrm{AdS}_3/\mathrm{CFT}_2$ at different points of the moduli space is a

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strong indication that in this system they are actually constant on the moduli space. This moduli-independence must be a consequence of the extended supersymmetry in $\mathcal{N} = (4,4)$ superconformal field theories, which implies a non-renormalization theorem for the 3-point functions of chiral primaries in theories of this type.

2.4 The bundle of chiral primaries and the chiral ring

In general comparing the correlation functions of operators at different points of the moduli space of a theory is not straightforward due to operator mixing. More precisely, to compare their correlation functions in a meaningful way, we first have to verify that the operators under comparison are actually "the same" at the two different points. Since the underlying quantum field theory is also changing as we vary the moduli, there is no natural identification of operators at different points of the moduli space. We could try to label operators by their conformal dimension and other conserved charges, but in general there is too large a degeneracy of operators of given charge to uniquely identify them. Moreover, as we will see later, the correct identification of operators between different points on the moduli space is actually path dependent.

Consider the moduli space $\mathcal{M}_{\mathrm{CFT}}$ of a CFT. At each point $p \in \mathcal{M}_{\mathrm{CFT}}$ we have the vector space $V_q^{(p)}$ of chiral primary operators of charge q. As we argued above we will assume that the dimension of this space is the same at all points, however there is no natural identification between the chiral primaries at different points of $\mathcal{M}_{\mathrm{CFT}}$. This means that $V_q^{(p)}$ is the fiber of a vector bundle

$$\mathcal{V}_q \tag{2.12}$$

of chiral primaries of charge q over the moduli space. The chiral ring coefficients can be thought of as multiplication between bundles of this form

$$C_{ij}^k : \mathcal{V}_p \otimes \mathcal{V}_q \to \mathcal{V}_{p+q} \tag{2.13}$$

and similarly the three point functions

$$C_{ij\overline{k}} : \mathcal{V}_p \otimes \mathcal{V}_q \otimes \overline{\mathcal{V}_{p+q}} \to \mathbb{C}. \tag{2.14}$$

It should be clear that to meaningfully compare the 3-point functions of chiral primaries at different points, we have to compute the connection on the bundles $\mathcal{V}_q$ which will specify how exactly we can "parallel transport" operators from one point to another. The connection on the bundle of operators over the moduli space is generally determined by the dynamics of the CFT as we explain in the next sections. In the special case of chiral primaries in theories with $\mathcal{N} = (2,2)$ supersymmetry this computation is simplified and the connection of the bundles $\mathcal{V}_q$ can be computed by the tt^* equations which will be described later.

In this paper we want to compute the geometry of the bundles of chiral primaries in $\mathcal{N} = (4,4)$ theories, and in particular for the theory relevant for $\mathrm{AdS}_3/\mathrm{CFT}_2$. The first result of our analysis is to show that the 3-point functions are covariantly constant, that is they satisfy

$$\nabla_\mu C_{ij\overline{k}} = 0 \tag{2.15}$$

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where ∇_μ is a covariant derivative[6] along a tangent direction on $\mathcal{M}_{\rm CFT}$, associated to the connection on the bundles $\mathcal{V}_q$. This is a non-renormalization theorem for the chiral primary 3-point functions in $\mathrm{AdS}_3/\mathrm{CFT}_2$ and more generally for any $\mathcal{N} = (4,4)$ theory. The second result is the computation of the connection on the bundles of chiral primaries at a general point on the moduli space of $\mathcal{N} = (4,4)$ theories, using the constraints from supersymmetry which allows us to express them in terms of the tt^* equations.

3 Families of conformal field theories and the connection for operators

The fact that we have to define a connection on the bundle of operators over the moduli space of a conformal field theory is quite general and not specific to theories with supersymmetry. The most familiar example is the connection for exactly marginal operators. The marginal operators $\mathcal{O}_\mu(z,\lambda)$ of a CFT correspond to tangent vectors on the moduli space at the point $\lambda \in \mathcal{M}_{\rm CFT}$. Comparing marginal operators at different points of $\mathcal{M}_{\rm CFT}$ is analogous to comparing tangent vectors at different points of a manifold, i.e. impossible, unless we first define a connection which describes their parallel transport. The moduli space $\mathcal{M}_{\rm CFT}$ of a conformal field theory has the structure of a Riemannian manifold. This structure is defined by the Zamolodchikov metric $g_{\mu\nu}(\lambda)$ which is given by the 2-point function

$$\langle \mathcal{O}_\mu(z,\lambda)\mathcal{O}_\nu(w,\lambda)\rangle = \frac{g_{\mu\nu}(\lambda)}{|z-w|^4}. \tag{3.1}$$

In general the metric $g_{\mu\nu}(\lambda)$ depends on the position $\lambda \in \mathcal{M}_{\rm CFT}$ which means that the moduli space has a non-trivial geometry. We can use the metric to define a metric-compatible connection for the operators $\mathcal{O}_\mu(z,\lambda)$, allowing us to parallel transport and compare them at different points of $\mathcal{M}_{\rm CFT}$. So the vector bundle of marginal operators is isomorphic to the tangent bundle of the moduli space and the natural connection on it is the Levi-Civita connection associated to the Zamolodchikov metric. The mixing of marginal operators under deformations of the theory is expressed by the equation

$$\delta_\mu \mathcal{O}_\nu = \Gamma^\kappa_{\nu\mu}\mathcal{O}_\kappa \tag{3.2}$$

where

$$\Gamma^\kappa_{\nu\mu} = \frac{1}{2} g^{\kappa\lambda}(\partial_\nu g_{\mu\lambda} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\mu\nu}) \tag{3.3}$$

and $g_{\mu\nu}(\lambda)$ is the Zamolodchikov metric defined in (3.1).[7]

[6] It should be clear that naive expression

$$\partial_\mu C_{ij\bar{k}} \stackrel{?}{=} 0 \tag{2.16}$$

is meaningless since the ordinary, instead of the covariant, derivative of a geometric object is not an invariant quantity.

[7] In general the marginal operators correspond to tangent vectors on the moduli space. The relation between operator mixing (3.2) and the Zamolodchikov metric (3.1) via (3.3) is true only if we choose a basis of marginal operators corresponding to commuting vector fields on the moduli space, so that they can be interpreted as derivatives with respect to a choice of coordinates. Otherwise they have to be treated in terms of a basis of vielbeins and the expression for their mixing has to be written in terms of the spin-connection.

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Similar arguments hold for operators of higher conformal dimension. For simplicity we can assume that at all points of $\mathcal{M}_{\rm CFT}$ we have a set of operators $\{\varphi_I\}$ of conformal weight $(h, \overline{h})$. If there are no additional conserved charges distinguishing them, then they will generically mix among themselves when we move on $\mathcal{M}_{\rm CFT}$. Under a deformation generated by a marginal operators $\mathcal{O}_\mu$ we have the mixing

$$\delta_\mu \varphi_I = A^J_{\mu I}\, \varphi_J \tag{3.4}$$

where $A^J_{\mu I}$ plays the role of the connection. Similarly if we consider an infinitesimal closed loop of deformations spanned by two marginal operators $\mathcal{O}_\mu, \mathcal{O}_\nu$, we have the curvature

$$(\delta_\mu \delta_\nu - \delta_\nu \delta_\mu)\, \varphi_I = R^J_{\mu\nu I}\, \varphi_J. \tag{3.5}$$

So the operators $\{\varphi_I\}$ take values in a vector bundle over the moduli space, whose connection is $A^J_{\mu I}$ and the curvature $R^J_{\mu\nu I}$. In what follows we will explain that there is a natural connection which is completely determined by the dynamics of the CFT.

3.1 Deformations of conformal field theories

Before we proceed, we would like to pause and discuss some (well-known) subtleties which will clarify the underlying reason for having a nontrivial connection for the operators in a family of conformal field theories. Let us start with a given theory characterized by a set of correlation functions

$$G_n(x) = \langle \varphi_1(x_1) \dots \varphi_n(x_n) \rangle \tag{3.6}$$

which satisfy the axioms of a 2-dimensional CFT.[8] We consider an operator $\mathcal{O}(z)$ in this theory. From the Lagrangian formulation point of view, we can deform the theory by adding to the action

$$S \to S + \frac{\lambda}{\pi} \int d^2 z \mathcal{O}(z) \tag{3.7}$$

where λ is a small parameter. The effect of this deformation is to modify the n-point functions

$$G_n(x) \to G_n(x) + \delta G_n(x). \tag{3.8}$$

For deformations of the form (3.7), the deformed n-point functions are given, to first order in λ, in terms of integrated $(n+1)$-point functions of the original undeformed theory

$$\delta G_n(x) \equiv \delta \langle \varphi_1(x_1) \dots \varphi_n(x_n) \rangle \simeq \frac{\lambda}{\pi} \int d^2 z \langle \varphi_1(x_1) \dots \varphi_n(x_n) \mathcal{O}(z) \rangle \tag{3.9}$$

where the meaning of the symbol $\simeq$ will become clear below. To second order in λ we have to consider the twice integrated $(n+2)$-point function of the undeformed theory and so on.

The deformed theory may be a local quantum field theory, but not necessarily a CFT. By demanding that the deformed correlation functions satisfy the CFT axioms, we find certain conditions for the deformation operator $\mathcal{O}(z)$. To first order in λ the condition is that $\mathcal{O}(z)$ must be an operator of dimension $(1,1)$, that is a marginal operator. More

[8] Notice that up to this point the correlation functions are defined only for distinct points, $x_i \neq x_j$.

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constraints from the requirement of conformal invariance appear at higher orders in λ, and if all these are satisfied $\mathcal{O}(z)$ is called an exactly marginal operator.

Going back to (3.9) we see that in order to compute the deformed correlators we have to integrate the insertion of $\mathcal{O}(z)$ over z, but when $z \to x_i$ the operator $\mathcal{O}(z)$ will hit the other insertions. This introduces two subtleties. First, in the original theory the correlators (3.6) were defined for distinct points, and formally we may have contact terms when the insertions coincide [12, 24]. Second, the integral over z in (3.9) will generally diverge because of short distance singularities between the operator $\mathcal{O}(z)$ and the other insertions $\varphi_i(x_i)$. So the right hand side of equation (3.9) is not well defined at this stage. Notice that for large z the correlator decays at least as $|z|^{-4}$, so there are no IR divergences to worry about.

Actually, the two aforementioned subtleties are related in the sense that we define the contact terms to precisely cancel the infinities arising from the integration over z around the punctures. While the infinities are cancelled in this way, there may be finite remaining contributions from this subtraction prescription which are responsible for the nontrivial connection for the operators of the CFT.

Equivalently we can forget about contact terms, but instead define a renormalization prescription for the integrated $(n+1)$-point function which is consistent with locality. Considering (3.9) again, we see that the more precise statement should be

$$\delta G_n(x) = \frac{\lambda}{\pi}\left[\int d^2z \langle \varphi_1(x_1)\dots\varphi_n(x_n)\mathcal{O}(z)\rangle\right]_{\text{ren}} \tag{3.10}$$

where the subscript *ren* stands for *renormalized*, and its exact meaning will be explained in the next subsection.

Now if we consider two deformations, one by the operator $\mathcal{O}_\mu$ and one by $\mathcal{O}_\nu$ then the naive answer (3.9) would give

$$(\delta_\mu\delta_\nu G_n)_{\text{naive}} \simeq \frac{\lambda_1\lambda_2}{\pi^2}\int d^2z_1\int d^2z_2 \langle \varphi_1(x_1)\dots\varphi_n(x_n)\mathcal{O}_\nu(z_1)\mathcal{O}_\mu(z_2)\rangle \tag{3.11}$$

and also

$$(\delta_\nu\delta_\mu G_n)_{\text{naive}} \simeq \frac{\lambda_1\lambda_2}{\pi^2}\int d^2z_1\int d^2z_2 \langle \varphi_1(x_1)\dots\varphi_n(x_n)\mathcal{O}_\mu(z_1)\mathcal{O}_\nu(z_2)\rangle \tag{3.12}$$

so formally

$$(\delta_\mu\delta_\nu G_n)_{\text{naive}} = (\delta_\nu\delta_\mu G_n)_{\text{naive}} \tag{3.13}$$

which would indicate that the order of deformation does not matter and it would imply that there is no curvature on the space of CFTs. However this is wrong, since the integrated $(n+2)$-point functions are not well defined for the reasons we mentioned earlier. Only the renormalized integrated $(n+2)$-point functions are meaningful

$$(\delta_\mu\delta_\nu G_n)_{\text{ren}} = \frac{\lambda_1\lambda_2}{\pi^2}\left[\int d^2z_1\int d^2z_2 \langle \varphi_1(x_1)\dots\varphi_n(x_n)\mathcal{O}_\nu(z_1)\mathcal{O}_\mu(z_2)\rangle\right]_{\text{ren}} \tag{3.14}$$

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where again we have to specify the way to renormalize the double integral. As it turns out, it is possible to find a renormalization prescription for the integrated correlation functions (3.10) and (3.14), such that the axioms of a CFT are preserved but the price we have to pay is that in general

$$(\delta_\mu \delta_\nu G_n)_{\text{ren}} \neq (\delta_\nu \delta_\mu G_n)_{\text{ren}}. \tag{3.15}$$

Because of this non-commutativity the correct statement is not

$$\delta_\mu G_n \stackrel{?}{=} \lambda \partial_\mu G_n \tag{3.16}$$

but rather

$$\delta_\mu G_n = \lambda \nabla_\mu G_n, \tag{3.17}$$

in other words

$$\nabla_\mu G_n = \frac{1}{\pi} \left[\int d^2 z \langle \varphi_1(x_1) \dots \varphi_n(x_n) \mathcal{O}_\mu(z) \rangle \right]_{\text{ren}} \tag{3.18}$$

The renormalization prescription defines the covariant derivative ∇_μ associated to the connection $A^J_{\mu I}$ on the vector bundle of the operators $\{\varphi_I\}$ introduced in (3.4).

3.2 The connection for operators

In [25, 26] connections on the vector bundle of operators over the moduli space of a CFT were studied in detail. A natural prescription (called the connection $\overline{c}$ in [26]) for defining the renormalized deformed correlators is the following: consider the to-be-integrated $(n+1)$-point function, introduce very small disks of size ϵ around the punctures x_i, and define the *regularized* integrated $(n+1)$-point function

$$\delta_\mu G_n(\epsilon) = \frac{\lambda}{\pi} \left[\int_{|z-x_i|>\epsilon} d^2 z \langle \varphi_1(x_1) \dots \varphi_n(x_n) \mathcal{O}_\mu(z) \rangle \right]_{\text{reg}}. \tag{3.19}$$

As $\epsilon \to 0$, and suppressing the x_i variables, the regularized integrated function will have the form

$$\delta_\mu G_n(\epsilon) = (\delta_\mu G_n)_{\text{ren}} + \sum_{\alpha>0} \frac{c_\alpha}{\epsilon^\alpha} + c_0 \log \epsilon \tag{3.20}$$

where the finite piece

$$(\delta_\mu G_n)_{\text{ren}} \tag{3.21}$$

defines the *renormalized* perturbed n-point function and the corresponding connection ∇_μ by (3.18).

If we consider the second variation of the correlation function according to this prescription, we find that

$$(\delta_\mu \delta_\nu G_n)_{\text{ren}} \neq (\delta_\nu \delta_\mu G_n)_{\text{ren}} \tag{3.22}$$

This is the reason that we have curvature on the vector bundles of operators over the moduli space.

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Also, notice that the vector bundle whose fiber is spanned by a set of operators $\{\varphi_I\}$ is equipped with a natural metric $g_{IJ}(\lambda)$ defined by the 2-point function

$$\langle \varphi_I(z)\varphi_J(w)\rangle = \frac{g_{IJ}(\lambda)}{(z-w)^{2h}(\overline{z}-\overline{w})^{2\overline{h}}}. \tag{3.23}$$

The connection defined above is compatible with the metric

$$\nabla_\mu g_{IJ} = 0 \tag{3.24}$$

so it is a natural connection for this vector bundle.

The curvature of the connection can be expressed in terms of 4-point functions. We quickly describe the main result, more details can be found in [26]. Consider a set of operators $\{\varphi_I\}$ of the same conformal dimension and same charges. The object we want to compute is the curvature $R^J_{\mu\nu I}$ of corresponding vector bundle over the moduli space. The curvature can be computed if we know the 4-point function

$$\langle \mathcal{O}_\mu(z_1)\mathcal{O}_\nu(z_2)\varphi_J(x_1)\varphi_I(x_2)\rangle \tag{3.25}$$

for distinct points of insertion. Generalizing the prescription (3.19), (3.20) the curvature is given by a twice integrated and appropriately regulated antisymmetrized combination of the 4-point function, as follows [26].

First we consider the 4-point function[9] as a function of z_1, z_2

$$G_{\mu\nu}(z_1,z_2) = \langle \mathcal{O}_\mu(z_1)\mathcal{O}_\nu(z_2)\varphi^J(\infty)\varphi_I(0)\rangle \tag{3.26}$$

for distinct points. Keeping z_1 fixed, we consider the integral over z_2 of the following expression[10]

$$F(z_1,\epsilon) = \frac{1}{\pi^2}\int_{\epsilon<|z_2|<1} d^2z_2 (G_{\mu\nu}(z_1,z_2) - G_{\nu\mu}(z_1,z_2)). \tag{3.27}$$

Notice that because of the antisymmetrization the integral converges as $z_1 \to z_2$. For fixed ϵ the integral is convergent, however it may diverge as $\epsilon \to 0$ because the operator at z_2 approaches the operator at 0. We define the regularized integral

$$\widetilde{F}(z_1) = \lim_{\epsilon\to 0}\left(F(z_1,\epsilon) - \mathrm{Dp}(F(z_1,\epsilon))\right) \tag{3.28}$$

where Dp denotes the divergent part, defined as in (3.20). This procedure gives us a finite function $\widetilde{F}(z_1)$. Finally we integrate $\widetilde{F}$ over z_1. There are divergences as $z_1 \to 0$ and again we are instructed to keep the finite part

$$R^J_{\mu\nu I} = \mathrm{Fp}\int_{|z_1|<1} d^2z_1 \widetilde{F}(z_1) \tag{3.29}$$

[9] The index J has been raised with the Zamolodchikov metric (3.23) as $\varphi^J = g^{JK}\varphi_K$.

[10] The integral over the disc arises from separating the plane, viewed as a two sphere, in two hemispheres, see [26].

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where Fp denotes the finite part, again defined as in (3.20). This is the final expression for the curvature. It is also possible to rewrite the curvature in terms of OPE coefficients. If we have

$$\mathcal{O}_\mu(z)\varphi_I(0) = \sum_k \frac{H^k_{\mu I}\varphi_k(0)}{z^{1+h_I-h_k}\overline{z}^{1+\overline{h}_I-\overline{h}_k}} \tag{3.30}$$

then after some algebra [26] we can show that the curvature has the form

$$R^J_{\mu\nu I} = 4\delta_{s_I,s_J}\left(\sum_{\gamma_k>\gamma_I} + \sum_{\gamma_k<\gamma_I}\right)\frac{H^k_{[\mu I}H^J_{\nu]k}\delta_{s_k,s_I}}{\gamma_{kJ}\gamma_{kI}} \tag{3.31}$$

where $\gamma = h + \overline{h}$, $s = h - \overline{h}$ are the scaling dimension and spin of the operator, and $\gamma_{ij} \equiv \gamma_i - \gamma_j$.

As we can see, the connection on the vector bundle of operators depends on the dynamics of the CFT. For a general interacting CFT it is difficult to compute the exact 4-point function, or equivalently the OPE coefficients, hence the computation of the curvature is hard. In theories with extended supersymmetry, and if we are interested in the curvature of operators in the chiral ring, it becomes possible to compute the curvature exactly. As we will see in this case the infinite sum in (3.31) truncates to a finite sum over chiral ring coefficients, giving us the tt^* equations. We analyze the $\mathcal{N} = (2,2)$ case in the next section and then consider $\mathcal{N} = (4,4)$ SCFTs.

4 The chiral ring of $\mathcal{N} = (2,2)$ theories

The bundle of chiral primaries has been analyzed in detail in theories with $\mathcal{N} = (2,2)$ superconformal symmetry. The main result relevant for us is the computation of the curvature of the bundle of chiral primaries in terms of the chiral ring coefficients, which is expressed by the tt^* equations derived by Cecotti and Vafa in [3]. In this section we quickly review the main points and give a derivation of the tt^* equations for superconformal theories which does not rely on the topological twisting.

In an $\mathcal{N} = (2,2)$ SCFT the left-moving currents are $T(z), G^\pm(z), J(z)$ and the right-moving ones $\overline{T}(\overline{z}), \overline{G}^\pm(\overline{z}), \overline{J}(\overline{z})$. The OPEs of the algebra can be found in appendix A. As we explained before in $\mathcal{N} = (2,2)$ theories we have the (cc) ring of chiral primary-chiral primary operators ϕ_i which satisfy

$$L_0 = \frac{J_0}{2}, \qquad \overline{L}_0 = \frac{\overline{J_0}}{2} \tag{4.1}$$

and their complex conjugates (aa) with opposite charges. We also have chiral primary - antichiral primary operators ψ_i in the (ca) ring satisfying

$$L_0 = \frac{J_0}{2}, \qquad \overline{L}_0 = -\frac{\overline{J_0}}{2} \tag{4.2}$$

and their complex conjugates in the (ac) ring. We will refer to the (cc) ring and its conjugate as the chiral ring, and to (ca) and its conjugate as the twisted chiral ring. The structure constants of the chiral ring are given by

$$\phi_i(z)\phi_j(w) = C^k_{ij}\phi_k(w) + \dots \tag{4.3}$$

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while those of the twisted chiral ring by

$$\psi_a(z)\psi_b(w) = \widetilde{C}^d_{ab}\psi_d(w) + \dots \tag{4.4}$$

We can find marginal operators by considering the descendants of chiral primaries of dimension $(1/2, 1/2)$. We have the following possibilities[11]

$$\mathcal{O}_i = \frac{1}{2}G^-_{-1/2}\overline{G^-}_{-1/2}\cdot\phi_i, \qquad \overline{\mathcal{O}_j} = \frac{1}{2}G^+_{-1/2}\overline{G^+}_{-1/2}\cdot\overline{\phi_j} \tag{4.5}$$

$$\mathcal{O}_a = \frac{1}{2}G^-_{-1/2}\overline{G^+}_{-1/2}\cdot\psi_a, \qquad \overline{\mathcal{O}_b} = \frac{1}{2}G^+_{-1/2}\overline{G^-}_{-1/2}\cdot\overline{\psi_b}. \tag{4.6}$$

All these are operators of conformal dimension $(1,1)$ and R-charge $(0,0)$, so they are marginal and can be used to perturb the CFT. The first class of operators labeled by $i, j, \dots$ are descendants of fields in the chiral ring and their complex conjugates, while the second class labeled by $a, b, \dots$ are descendants of fields in the twisted chiral ring and their complex conjugates. We use Greek indices $\mu, \nu, \dots$ to denote a general marginal operator which can be of any of the four forms described above.

A basic result is that for $\mathcal{N} = (2,2)$ SCFTs the moduli space locally has a product structure

$$\mathcal{M}_{\rm CFT} = \mathcal{M}_C \times \mathcal{M}_{\rm TC} \tag{4.7}$$

where $\mathcal{M}_C$ is generated by marginal operators which come from the chiral ring, and $\mathcal{M}_{\rm TC}$ is generated by marginal operators from the twisted chiral ring. As an example, for a sigma-model whose target space is a Calabi-Yau 3-fold, one of the spaces corresponds to the Kähler structure deformations while the other to the complex structure deformations. It can be shown that each of the two components $\mathcal{M}_C, \mathcal{M}_{\rm TC}$ is a complex, Kähler manifold. Moreover it can be shown that they are special Kähler.

We denote by $g_{i\bar{j}}$ the Kähler form of the component $\mathcal{M}_C$ and $g_{a\bar{b}}$ that of $\mathcal{M}_{\rm TC}$, which are given in terms of CFT data by the two point functions

$$\langle\mathcal{O}_i(z)\overline{\mathcal{O}_j}(w)\rangle = \frac{g_{i\bar{j}}}{|z-w|^4}, \qquad \langle\mathcal{O}_a(z)\overline{\mathcal{O}_b}(w)\rangle = \frac{g_{a\bar{b}}}{|z-w|^4}. \tag{4.8}$$

In $\mathcal{N} = (4,4)$ theories the moduli space does not factorize, not even locally. It consists of a single factor and cannot be decomposed into chiral and twisted chiral components. As we will see it is not a complex manifold.

4.1 Curvature of the algebra

We now proceed with a discussion of the connection on the bundle of operators over the moduli space. In the same way that chiral primaries can mix under deformations of the CFT, the generators of the algebra can also mix among themselves, see [27] for a nice review. The energy momentum tensor $T(z)$ and the U(1) current $J(z)$ are uniquely defined at each point of the moduli space, so there can be no holonomy associated to them. However the

[11] We have included the factors of $\frac{1}{2}$ in the normalization of the marginal operators to ensure that $g_{i\bar{j}} \equiv \langle\mathcal{O}_i(1)\overline{\mathcal{O}_j}(0)\rangle = \langle\phi_i(1)\overline{\phi_j}(0)\rangle$.

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supercurrents are not uniquely defined, since the $\mathcal{N}=(2,2)$ algebra has a $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ automorphism which transforms the supercurrents as

$$G^{\pm} \to e^{\pm i\theta} G^{\pm}, \qquad \overline{G}^{\pm} \to e^{\pm i\overline{\theta}} \overline{G}^{\pm} \tag{4.9}$$

leaving the bosonic currents unchanged, and where $\theta, \overline{\theta}$ are two independent angles. Consequently, what we mean by a supercurrent is ambiguous up to an overall phase. Moreover, if we parallel transport on the moduli space and come back to the original point, the supercurrents will receive a U(1) rotation. This means that the supercurrents are (operator valued) sections of U(1) bundles over the moduli space. If G^+ is a section of a U(1) bundle $\mathcal{L}$ then G^- will be a section of $\mathcal{L}^{-1}$, since they transform with opposite phases. Similarly $\overline{G}^+$ will be a section of another bundle $\overline{\mathcal{L}}$ and $\overline{G}^-$ a section of $\overline{\mathcal{L}}^{-1}$. We call F the curvature tensor of the bundle $\mathcal{L}$ and $\overline{F}$ the curvature of $\overline{\mathcal{L}}$. According to our previous discussion, to compute the curvature of $\mathcal{L}$ and $\overline{\mathcal{L}}$, we need the 4-point functions

$$\langle \mathcal{O}_\mu(x)\mathcal{O}_\nu(y) G^r(z) G^s(w)\rangle \quad \text{and} \quad \langle \mathcal{O}_\mu(x)\mathcal{O}_\nu(y) \overline{G}^r(z) \overline{G}^s(w)\rangle, \tag{4.10}$$

where $\mathcal{O}_\mu, \mathcal{O}_\nu$ are marginal operators of the form (4.5),(4.6) and $r, s = \pm$.

These four point functions can be exactly computed using the superconformal Ward identities of the $\mathcal{N}=(2,2)$ algebra. For example as we show in appendix C we have

$$\langle \mathcal{O}_i(x)\overline{\mathcal{O}_j}(y) G^+(z) G^-(w)\rangle = \frac{2c}{3}\frac{g_{i\overline{j}}}{|x-y|^4(z-w)^3} + \frac{2g_{i\overline{j}}}{(x-z)^2(y-w)^2(z-w)(\overline{x}-\overline{y})^2} \tag{4.11}$$

and similarly for the other combinations. Following the prescription of equations (3.26) to (3.29) we find that the only nonzero components of the curvature for the line bundle $\mathcal{L}$ are

$$F_{i\overline{j}} = -\frac{3}{c} g_{i\overline{j}}, \quad F_{a\overline{b}} = -\frac{3}{c} g_{a\overline{b}} \tag{4.12}$$

while for $\overline{\mathcal{L}}$ we have

$$\overline{F}_{i\overline{j}} = -\frac{3}{c} g_{i\overline{j}}, \quad \overline{F}_{a\overline{b}} = \frac{3}{c} g_{a\overline{b}} \tag{4.13}$$

Notice that if we consider the bundle $\mathcal{L} \otimes \overline{\mathcal{L}}$, then its curvature is zero on $\mathcal{M}_{\mathrm{TC}}$, while $\mathcal{L} \otimes \overline{\mathcal{L}}^{-1}$ has zero curvature over $\mathcal{M}_C$.

To summarize, we found that while the bosonic currents $T(z), J(z)$ are well defined everywhere, the supercurrents $G^{\pm}(z)$ are ambiguous and there is an associated holonomy for them described by the holomorphic line bundles $\mathcal{L}$, $\overline{\mathcal{L}}$ over the moduli space. Notice that the Kähler form on the moduli space is $\frac{c}{3}$ times the curvature of the line bundle $\mathcal{L}$ (or $\overline{\mathcal{L}}$), so its first Chern class is $\frac{c}{3}$ times an integral class [28, 29]. For sigma-models in Calabi-Yau n-folds, where $c = 3n$, the bundle $\mathcal{L}^{c/3}$ is the same as the line bundle of the holomorphic $(n,0)$ form Ω over the complex structure moduli space.

4.2 On the curvature of the chiral primaries

Now we want to consider the connection on the bundle of chiral primaries. From charge conservation, (cc) operators can only mix with themselves, and similarly for $(aa), (ca), (ac)$.

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For each conformal dimension $(h, \overline{h})$ we have the bundle of chiral primaries ϕ_i with charge $(2h, 2\overline{h})$, the bundle of twisted chiral primaries with charge $(2h, -2\overline{h})$ and their hermitian conjugates.

To avoid overly heavy notation we will denote the total bundle of chiral primaries by $\mathcal{V}$ and that of twisted chiral primaries by $\widetilde{\mathcal{V}}$. Each of these bundles is the direct sum of subbundles $\mathcal{V}_q$ corresponding to fields of specific charges

$$\mathcal{V} = \sum_q \oplus \mathcal{V}_q \tag{4.14}$$

It should be clear that the connection on the bundle $\mathcal{V}$ has to preserve the grading by conformal dimension (or U(1) charge), since it should not mix operators of different dimensions under parallel transport.

4.3 Direct computation of the curvature of chiral primaries

There are two methods to compute the curvature of chiral primaries: one is to directly compute the relevant 4-point function in the physical theory and then use (3.29). The second is to use spectral flow to the Ramond sector, consider the topologically twisted theory and follow the arguments of [3]. The two methods give the same result, which is the tt^* equations. In this section we show how the direct computation of the 4-point function yields an alternative derivation of the tt^* equations in superconformal theories.

Let us consider the curvature of the bundle $\mathcal{V}$ over the factor $\mathcal{M}_C$ of the moduli space. According to the general expression (3.29), we need to compute the 4-point functions

$$\langle \mathcal{O}_i(x)\overline{\mathcal{O}_j}(y)\phi_k(z)\overline{\phi_l}(w)\rangle, \qquad \langle \overline{\mathcal{O}_j}(x)\mathcal{O}_i(y)\phi_k(z)\overline{\phi_l}(w)\rangle \tag{4.15}$$

where

$$\mathcal{O}_i(x) = \frac{1}{2}G^-_{-1/2}\overline{G}^-_{-1/2}\cdot\phi_i(x), \qquad \overline{\mathcal{O}_j}(y) = \frac{1}{2}G^+_{-1/2}\overline{G}^+_{-1/2}\cdot\overline{\phi_j}(y). \tag{4.16}$$

As explained in appendix D, using the OPEs of the supercurrents with the chiral primaries, we can move the supercurrent operators from $\mathcal{O}_i$ onto $\mathcal{O}_{\overline{j}}$ and we have

$$\langle \mathcal{O}_i(x)\overline{\mathcal{O}_j}(y)\phi_k(z)\overline{\phi_l}(w)\rangle = \partial_y\partial_{\overline{y}}\left(\frac{|y-z|^2}{|x-z|^2}\langle\phi_i(x)\overline{\phi_j}(y)\phi_k(z)\overline{\phi_l}(w)\rangle\right) \tag{4.17}$$

similarly moving the supercurrents from $\mathcal{O}_{\overline{j}}$ to $\mathcal{O}_i$ we have

$$\langle \overline{\mathcal{O}_j}(x)\mathcal{O}_i(y)\phi_k(z)\overline{\phi_l}(w)\rangle = \partial_y\partial_{\overline{y}}\left(\frac{|y-w|^2}{|x-w|^2}\langle\overline{\phi_j}(x)\phi_i(y)\phi_k(z)\overline{\phi_l}(w)\rangle\right) \tag{4.18}$$

Taking $w \to 0$ and $z \to \infty$ we find

$$\langle \mathcal{O}_i(x)\overline{\mathcal{O}_j}(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle = \partial_y\partial_{\overline{y}}\left(\langle\phi_i(x)\overline{\phi_j}(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle\right) \tag{4.19}$$

and

$$\langle \overline{\mathcal{O}_j}(x)\mathcal{O}_i(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle = \partial_y\partial_{\overline{y}}\left(\frac{|y|^2}{|x|^2}\langle\overline{\phi_j}(x)\phi_i(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle\right) \tag{4.20}$$

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We can now use the general formula (3.29) for the curvature of the bundles of operators. We notice that as $y \to 0$ both of the correlation functions are finite: for (4.19) we just have to use the OPE in the antichiral ring which is non-singular, while for (4.20) we have to use the results from appendix E for the OPE of a chiral field with an antichiral. The leading term goes like $\frac{1}{|y|^2}$ and is exactly cancelled by the $|y|^2$ in the numerator. Following (3.29) the curvature is

$$R_{i\bar{j}} = \mathrm{Fp}\frac{1}{(\pi)^2}\int_{|x|<1} d^2x\, I(x) \tag{4.21}$$

where

$$\begin{aligned} I(x) = \int_{|y|<1} d^2y\, \partial_y\partial_{\overline{y}} \left(\langle \phi_i(x)\overline{\phi_j}(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle\right) \\ - \partial_y\partial_{\overline{y}}\left(\frac{|y|^2}{|x|^2}\langle\overline{\phi_j}(x)\phi_i(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle\right) \end{aligned} \tag{4.22}$$

and we used the fact that there is no singularity as $y \to 0$. Using Gauss's theorem we have[12]

$$I(x) = \frac{1}{4}\int_{|y|=1} d\theta_1 (y\partial_y + \overline{y}\partial_{\overline{y}})\left(\langle \phi_i(x)\overline{\phi_j}(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle - \frac{|y|^2}{|x|^2}\langle\overline{\phi_j}(x)\phi_i(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle\right) \tag{4.23}$$

From the conformal Ward identity

$$\sum_i (h_i + z_i\partial_i)\langle\varphi_1(z_1)\dots\varphi_n(z_n)\rangle = 0 \tag{4.24}$$

we have for the 4-point function

$$(1 + x\partial_x + y\partial_y)\langle \phi_i(x)\overline{\phi_j}(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle = 0 \tag{4.25}$$

Using this we can write (4.23) as

$$I(x) = -\frac{1}{4}\int_{|y|=1} d\theta_1 (2 + x\partial_x + \overline{x}\partial_{\overline{x}})\left(\langle \phi_i(x)\overline{\phi_j}(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle - \frac{1}{|x|^2}\langle\overline{\phi_j}(x)\phi_i(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle\right) \tag{4.26}$$

Considering the integration over x we find

$$\begin{aligned} R_{i\bar{j}} = -\frac{1}{(2\pi)^2}\int_0^1 dr \int_{|y|=1} d\theta_1 \int_{|x|=r} d\theta_2 \frac{d}{dr}\Big(r^2\langle \phi_i(x)\overline{\phi_j}(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle \\ -\langle\overline{\phi_j}(x)\phi_i(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle\Big) \end{aligned} \tag{4.27}$$

So we have

$$\begin{aligned} R_{i\bar{j}} = &-\frac{1}{(2\pi)^2}\lim_{|r|\to 1}\int_{|y|=1} d\theta_1 \int_{|x|=r} d\theta_2 \Big(r^2\langle \phi_i(x)\overline{\phi_j}(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle \\ &\qquad - \langle\overline{\phi_j}(x)\phi_i(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle\Big) \\ &+\frac{1}{(2\pi)^2}\lim_{|r|\to 0}\int_{|y|=1} d\theta_1 \int_{|x|=r} d\theta_2 \Big(r^2\langle \phi_i(x)\overline{\phi_j}(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle \\ &\qquad - \langle\overline{\phi_j}(x)\phi_i(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle\Big) \end{aligned} \tag{4.28}$$

[12] As explained in [26] the antisymmetrized 4-point function has no singularity as $y \to x$.

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The contribution from the first two terms can be computed using the OPE between ϕ_i and $\overline{\phi_j}$ as explained in appendix F. The contribution form the second term can be computed using the OPE of the field at x with the field at 0, which is determined by the chiral ring coefficients (see also appendix E). Finally we have

$$\begin{aligned} R_{i\bar{j}} &= g_{i\bar{j}} g_{k\bar{l}} \left(1 - \frac{3}{c}(q+\overline{q})\right) - C^m_{ik} g_{m\overline{n}} C^{*\overline{n}}_{\overline{j}\overline{l}} + g_{k\overline{m}} C^{*\overline{m}}_{\overline{j}\overline{n}} g^{\overline{n}\rho} C^{\sigma}_{i\rho} g_{\sigma\overline{k}} \\ &= g_{i\bar{j}} g_{k\bar{l}} \left(1 - \frac{3}{c}(q+\overline{q})\right) - [C_i, \overline{C_j}] \end{aligned} \tag{4.29}$$

All other components of the curvature vanish, as can be easily demonstrated using a similar analysis. To summarize we find the following expressions for the curvature

$$\begin{aligned} &[\nabla_i, \nabla_j] = 0 \\ &[\nabla_{\bar{i}}, \nabla_{\bar{j}}] = 0 \\ &[\nabla_i, \nabla_{\bar{j}}] = g_{i\bar{j}} g_{k\bar{l}} \left(1 - \frac{3}{c}(q+\overline{q})\right) - [C_i, \overline{C_j}] \end{aligned} \tag{4.30}$$

Apart from the term proportional to $g_{i\bar{j}} g_{k\bar{l}}$ in the third equation, these are the tt^* equations which were initially derived [3] using the correspondence between chiral primaries in the NS sector and the Ramond ground states, and the topological twisting of theories with extended supersymmetry. More details can be found in the relevant papers. While the derivation based on the topological twisting is more general, as it also works for non-conformal $\mathcal{N} = (2,2)$ theories, it is satisfying that the same result can be reproduced from the point of view of conformal perturbation theory in the physical theory without using the twisting. We discuss the role of the extra term in the next subsection.

The main use of these equations is that for $\mathcal{N} = (2,2)$ theories we can compute the connection on the bundles of chiral primaries if we know the chiral ring coefficients. In general the chiral ring coefficients are not constant, rather they are holomorphic functions on the moduli space. Later we will see the simplifications that occur for $\mathcal{N} = (4,4)$ theories.

Before we proceed let us mention that similarly we can compute the curvature of the bundle of the twisted chiral ring $\widetilde{\mathcal{V}}$ over the factor $\mathcal{M}_{\mathrm{TC}}$ of the moduli space and we similarly find the equation

$$R_{a\bar{b}} = g_{a\bar{b}} g_{c\bar{d}} \left(1 - \frac{3}{c}(q-\overline{q})\right) - [\widetilde{C}_a, \overline{\widetilde{C}_b}]. \tag{4.31}$$

4.4 Some comments

In the original tt^* equations for the Ramond ground states of the topologically twisted theory, the term

$$g_{i\bar{j}} g_{k\bar{l}} \left(1 - \frac{3}{c}(q+\overline{q})\right) \tag{4.32}$$

was not present. This means that the connection for the Ramond states in the topologically twisted theory is not exactly the same as the connection for NS chiral primaries in the physical theory, but they differ by U(1) phases related to the line bundles $\mathcal{L}, \overline{\mathcal{L}}$. While this extra term came out of our computation naturally, using the general formalism for the

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connection of operators, we have not fully understood why there is a difference between the physical and twisted theories. Because of this we would like to make some consistency checks regarding the presence of this term. In this section we will consider a special class of chiral primaries and we will see that to get the correct answer for their curvature we do indeed need the extra term (4.32).

First we consider the case of the identity operator $I(z)$ whose charges are $(0,0)$. Obviously its curvature over the moduli space should be zero. This can be seen from the 4-point function

$$\langle \mathcal{O}_i(x)\overline{\mathcal{O}_j}(y)I(z)I(w)\rangle = \frac{g_{i\overline{j}}}{|x-y|^4} \tag{4.33}$$

This is symmetric under $x \leftrightarrow y$, so its curvature must vanish. Now, if we compute the second term of (4.30) on the subspace spanned by $I(z)$ we have

$$[C_i, \overline{C_j}] = g_{i\overline{j}} \tag{4.34}$$

This is precisely cancelled by the term (4.32) for $q = \overline{q} = 0$.

Another example we will consider is the chiral primary $\rho(z)$ of highest left U(1) charge $(c/3, 0)$. This is a unique field present in any $\mathcal{N} = (2,2)$ theory. To compute the relevant 4-point function we consider the bosonization of the U(1) currents

$$J(z) = i\sqrt{c/3}\ \partial H, \qquad \overline{J}(\overline{z}) = i\sqrt{c/3}\ \overline{\partial}\,\overline{H} \tag{4.35}$$

where H, $\overline{H}$ are free compact bosons. Any operator φ with charge $(q, \overline{q})$ can be written as

$$\varphi = e^{i\sqrt{3/c}(qH+\overline{q}\overline{H})}\chi \tag{4.36}$$

with χ a neutral operator, which may be a polynomial in $J \sim \partial H$ and $\overline{J} \sim \overline{\partial H}$. The field $\rho(z)$ has charges $(c/3, 0)$ and using the bosonized currents can be written as

$$\rho(z) = e^{i\sqrt{c/3}H} \tag{4.37}$$

The marginal operators are neutral so if we write them in the form (4.36) then the H-dependence can be at most a polynomial in derivatives of the fields $H, \overline{H}$, or equivalently polynomial in the currents $J, \overline{J}$ and their derivatives. However we know that for the marginal operators which are descendants of chiral primaries we have

$$J(z)\mathcal{O}(w) = \text{regular} \tag{4.38}$$

which means that actually these marginal operators do not involve the free boson $H(z)$ at all (similarly for the right moving $\overline{H}$). But this implies that

$$\rho(z)\mathcal{O}(w) = \text{regular} \tag{4.39}$$

Now we consider the 4-point function

$$\langle O_i(x)\overline{O_j}(y)\rho(z)\rho^\dagger(w)\rangle \tag{4.40}$$

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The field $\rho(z)$ is holomorphic so we can compute the 4-point function from the OPEs. From (4.39) we see that the only nontrivial OPE is between $\rho(z)$ and $\rho^\dagger(w)$ which is of the form

$$\rho(z)\rho^\dagger(w) = \frac{g_{\rho\overline{\rho}}}{(z-w)^{2c/3}} + \dots \tag{4.41}$$

where the operators appearing in the dots only involve the free boson H. As we argued the marginal operators do not couple to H, so the 4-point function is equal to

$$\langle O_i(x)\overline{O_j}(y)\rho(z)\rho^\dagger(w)\rangle = \frac{g_{i\overline{j}}g_{\rho\overline{\rho}}}{|x-y|^4(z-w)^{2c/3}} \tag{4.42}$$

Again this is symmetric in $x \leftrightarrow y$ so the curvature of the field $\rho(z)$ should vanish. Looking at (4.30) we find that for this field

$$[C_i, \overline{C_j}] = 0 \tag{4.43}$$

while the term (4.32) is also zero for $q = c/3, \overline{q} = 0$. So indeed the curvature vanishes. Similarly one can study the right moving field $\overline{\rho}$ of charge $(0, c/3)$. Finally we consider the field $\mathcal{A} = (\rho\overline{\rho})$ of charge $(c/3, c/3)$. Since this is the product of ρ and $\overline{\rho}$ its curvature should also vanish. The second term of (4.30) for this field gives

$$[C_i, \overline{C_j}] = -g_{i\overline{j}}g_{\mathcal{A}\overline{\mathcal{A}}} \tag{4.44}$$

This is precisely cancelled by the term (4.32) for $q = \overline{q} = c/3$.

The conclusion is that in all these cases the presence of the term $g_{i\overline{j}}g_{k\overline{l}}\left(1 - \frac{3}{c}(q+\overline{q})\right)$ is necessary to give the correct answer for the curvature of the operators. See also footnote (21) for some related observations.

Notice that the extra term is reminiscent of duality between $H^{p,q}(M)$ and $H^{d-p,d-q}(M)$ for a $2d$ dimensional Calabi-Yau manifold. Perhaps its presence/absence is related to whether one uses the standard basis for the chiral primaries, (which means that they are directly related to the Dolbeault cohomology in the case of a supersymmetric sigma model), or a dual basis. It would be interesting to explore this a bit further.

5 The $\mathcal{N} = (4,4)$ superconformal algebra

In this section we review some basic properties of the (small) $\mathcal{N} = (4,4)$ superconformal algebra, whose OPEs can be found in appendix A. Its R-symmetry group is $\mathrm{SO}(4)^R = \mathrm{SU}(2)^R_{\text{left}} \times \mathrm{SU}(2)^R_{\text{right}}$. The left-moving currents are the energy momentum tensor T and the currents of the $\mathrm{SU}(2)^R_{\text{left}}$ symmetry $J_i, i = 1,2,3$. The left-moving supercurrents fall into two doublets of the $\mathrm{SU}(2)^R_{\text{left}}$ and will be denoted by $G^{a,i}$, $a, i = 1,2$, obeying a reality condition $G^{a,i} = \epsilon^{ab}\epsilon^{ij}(G^{b,j})^*$. The $\mathrm{SU}(2)^R_{left}$ acts on the a index. The level of the $\mathrm{SU}(2)^R_{left}$ current algebra is equal to $k = \frac{c}{6}$, where c is the central charge of the theory. We have the same structure on the right-moving sector and we denote the right-moving generators by $\overline{T}, \overline{J}_i$ and $\overline{G}^{a,i}$.

The $\mathcal{N} = (4,4)$ algebra has an outer automorphism which rotates the supercurrents, leaving all bosonic generators unchanged. In the notation $G^{a,i}$ for the supercurrents the outer automorphism is SU(2) rotations of the i-index. In general this transformation is

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not a symmetry of the theory, as there is no corresponding conserved current generating it. We will call it $\mathrm{SO}(4)^{\mathrm{outer}} = \mathrm{SU}(2)^{\mathrm{outer}}_{\mathrm{left}} \times \mathrm{SU}(2)^{\mathrm{outer}}_{\mathrm{right}}$. We remind that the $\mathrm{SO}(4)^R$ symmetry rotates both the supercurrents and the R-currents $J_i, \overline{J}_i$, while $\mathrm{SO}(4)^{\mathrm{outer}}$ rotates only the supercurrents. The full automorphism group of the algebra is $G = \mathrm{SO}(4)^R \times \mathrm{SO}(4)^{\mathrm{outer}}$.

5.1 $\mathcal{N} = (2,2)$ subalgebras

An $\mathcal{N} = (4,4)$ theory can of course be also seen as $\mathcal{N} = (2,2)$. To pick an $\mathcal{N} = (2,2)$ subalgebra of the $\mathcal{N} = (4,4)$ we have to do two things. First we have to choose a Cartan generator of $\mathrm{SU}(2)^R_{\mathrm{left}}$ and one of $\mathrm{SU}(2)^R_{\mathrm{right}}$ that we will identify with the U(1) R-charge of the $\mathcal{N} = (2,2)$ theory. This gives us a freedom of $\left(\frac{\mathrm{SU}(2)}{\mathrm{U}(1)}\right) \times \left(\frac{\mathrm{SU}(2)}{\mathrm{U}(1)}\right)$. Notice that the different choices can be related by an $\mathrm{SO}(4)^R$ transformation which is a symmetry of the theory, so they are essentially equivalent. Second, after we pick the direction of the $\mathcal{N} = (2,2)$ R-charge generators, we still have an extra $\left(\frac{\mathrm{SU}(2)}{\mathrm{U}(1)}\right) \times \left(\frac{\mathrm{SU}(2)}{\mathrm{U}(1)}\right)$ freedom to choose which combination of the supercurrents $G^{\pm+}, G^{\pm-}$ will be identified as the "standard" supercurrents of the $\mathcal{N} = (2,2)$ theory. The different choices of the supercurrents are related by the outer automorphism $\mathrm{SO}(4)^{\mathrm{outer}}$ which is not a symmetry, so in general the different $\mathcal{N} = (2,2)$ subalgebras of this type will be inequivalent.

Notice that once we make the first choice and orient the $\mathrm{U}(1) \times \mathrm{U}(1)$ generators in the $\mathrm{SO}(4)^R$, we completely fix which operators we will call chiral primaries (the operators with $(L_0, \overline{L}_0) = (J^3_0, \overline{J}^3_0)$), independent of the remaining ambiguity in the choice of the supercurrents. This is a consequence of the fact that for a superconformal primary the following conditions are equivalent[13]

$$(L_0 - J^3_0)|\phi\rangle = 0 \quad \Leftrightarrow \quad G^{++}_{-1/2}|\phi\rangle = 0 \quad \Leftrightarrow \quad G^{+-}_{-1/2}|\phi\rangle = 0 \tag{5.1}$$

Even though the definition of a chiral primary does not depend on the choice of the supercurrents, its descendants do depend on it. So the inequivalent $\mathcal{N} = (2,2)$ subalgebras with the same $\mathrm{SO}(4)^R$ orientation but with different $\mathrm{SO}(4)^{\mathrm{outer}}$ orientation have the same chiral primaries, but different descendants.

5.2 Short representations

In this section we describe the short representations of the $\mathcal{N} = 4$ algebra, i.e. those which saturate the BPS bound [30] . For simplicity we will only discuss the representation on the left-moving sector. To get a full representation of the $\mathcal{N} = (4,4)$ algebra we have to tensor a left with a right-moving representation. Short representations can be constructed by starting with a chiral primary field and then acting on it with the generators of the algebra. The conformal dimension and R-charge of a chiral primary satisfy

$$L_0|\phi\rangle = J^3_0|\phi\rangle = q|\phi\rangle \tag{5.2}$$

[13]Notice the difference in conventions between the normalization of the R-charge for the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ cases. In the $\mathcal{N} = 2$ theories, the U(1) charge J is normalized to take integral values and the BPS bound is $L_0 = J_0/2$. In the $\mathcal{N} = 4$ conventions, which we are going to follow in the rest of this paper, the eigenvalues of J^3_0 are half-integers and the BPS bound is $L_0 = J^3_0$.

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We use the notation $(L_0, J_0^3) = (q, q)$ for the conformal dimension and J^3 charge. Such a field is annihilated by the supercurrents

$$G^{++}_{-1/2}|\phi\rangle = G^{+-}_{-1/2}|\phi\rangle = 0 \tag{5.3}$$

To construct the representation we first discuss the action of $G^{a,i}_{-1/2}$ and J_0^i on the highest weight state. To start, we can act on ϕ with the lowering operator J_0^{--} with respect to the J_0^3 charge. This gives us the fields $(J^{--})^n\phi$ with quantum numbers $(L_0, J_0^3) = (q, q-n)$. Obviously we can act at most $2q$ times before the state is annihilated. This set of fields forms a $2q+1$ dimensional spin-q representation of $\mathrm{SU}(2)^R_{\mathrm{left}}$, and they all have the same conformal dimension.[14] Also notice that these states are singlets of the $\mathrm{SU}(2)^{\mathrm{outer}}_{\mathrm{left}}$.

We can construct more states of the representation by acting on ϕ with one supercurrent. The only supercurrents that do not annihilate ϕ are G^{-+}, G^{--} which mix under the action of $\mathrm{SU}(2)^{\mathrm{outer}}_{\mathrm{left}}$. This way we get two states

$$|\psi^+\rangle = G^{-+}_{-1/2}|\phi\rangle, \quad |\psi^-\rangle = G^{--}_{-1/2}|\phi\rangle \tag{5.4}$$

These states have charges equal to $(L_0, J_0^3) = (q+\frac{1}{2}, q-\frac{1}{2})$ and they are a doublet of the $\mathrm{SU}(2)^{\mathrm{outer}}_{\mathrm{left}}$. Acting on these states with J^{--} we can complete them into spin $q-\frac{1}{2}$ representation of $\mathrm{SU}(2)^R_{\mathrm{left}}$.

Finally we can get new states acting on ϕ with two supercurrents. This gives the state

$$|\Phi\rangle = G^{-+}_{-1/2}G^{--}_{-1/2}|\phi\rangle \tag{5.5}$$

It has $(L_0, J_0^3) = (q+1, q-1)$. It is a singlet of $\mathrm{SU}(2)^{\mathrm{outer}}_{\mathrm{left}}$. Acting on this state with J^{--} we generate a spin $q-1$ representation of $\mathrm{SU}(2)^R_{\mathrm{left}}$.

The full representation of the superconformal algebra is generated by taking conformal descendants of the states described above. This is the structure of the typical short representation. If we start with a chiral primary of low enough conformal dimension we get special short representations that we review in the next subsection.

5.3 Special short representations

First we consider the shortest nontrivial representation. If we start with a chiral primary with $(L_0, J_0^3) = (\frac{1}{2}, \frac{1}{2})$ and act with the supercurrents G^{-+}, G^{--} we get two states with $(L_0, J_0^3) = (1, 0)$. We cannot act again with the supercurrents since it would give a negative value for the R-charge. The representation is terminated and is shorter than the typical short representation. The two fields $|\psi^+\rangle = G^{-+}_{-1/2}|\phi\rangle$, $|\psi^-\rangle = G^{--}_{-1/2}|\phi\rangle$ are singlets of the $\mathrm{SU}(2)^R_{\mathrm{left}}$ and a doublet of $\mathrm{SU}(2)^{\mathrm{outer}}_{\mathrm{left}}$. If we tensor them with a similar representation from the right-moving sector we get fields with conformal dimension $(L_0, \overline{L}_0) = (1, 1)$ which are singlets of $\mathrm{SO}(4)^R$, but which transform under $\mathrm{SO}(4)^{\mathrm{outer}}$. These are the marginal operators of the theory.

[14] We called the top component ϕ chiral primary, but each of the fields $(J^{--})^n\phi$ would also be "chiral primary" under a different orientation of the J^3 axis.

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Second let us consider the case where we start with a chiral primary $|\phi\rangle$ with $(L_0, J_0^3) = (1,1)$. From the previous analysis we see that the state $|\Phi\rangle = G^{-+}_{-1/2} G^{--}_{-1/2}|\phi\rangle$ has $(L_0, J_0^3) = (2,0)$. It is a singlet of $\mathrm{SU}(2)^R_{\mathrm{left}}$ and also a singlet of $\mathrm{SU}(2)^{\mathrm{outer}}_{\mathrm{left}}$. If we tensor it with a similar representation from the right-moving sector we get fields which have conformal dimension $(L_0, \overline{L}_0) = (2,2)$ and are singlets of the $\mathrm{SO}(4)^R$ (and also singlets of $\mathrm{SO}(4)^{\mathrm{outer}}$). These fields are the leading irrelevant operators which are singlets under $\mathrm{SO}(4)^R$, so they break the conformal invariance but not the $\mathcal{N} = (4,4)$ supersymmetry. Notice that there are no other $\mathrm{SO}(4)^R$ singlet operators in the short multiplets of the algebra.

5.4 The moduli space of $\mathcal{N} = (4,4)$ SCFTs

Let us now use the restrictions of the $\mathcal{N} = (4,4)$ superconformal symmetry on the structure of the moduli space. We review the well-known argument which completely determines the local structure of the moduli space of any $\mathcal{N} = (4,4)$ SCFT [12, 13].

As we saw before, motion on the moduli space is generated by descendants of chiral primaries with $(q, \overline{q}) = (\frac{1}{2}, \frac{1}{2})$. Let us say that there are n multiplets of this form. Each multiplet gives 4 real marginal operators so the dimension of the moduli space will be $4n$. The (local) holonomy on this space is in general $\mathrm{SO}(4n)$. However the marginal operators come in groups of 4 from a single chiral primary. We want to take advantage of this fact to restrict the holonomy of the moduli space. The chiral primaries ϕ_i of weight $(\frac{1}{2}, \frac{1}{2})$ are sections of a vector bundle and have themselves some holonomy. Also, to go from the chiral primaries to the moduli, we have to act with the supercurrents. This means that the marginal operators are sections of a bundle which is the tensor product of the bundle of the chiral primaries with the bundle of the supercurrents. So the holonomy on the tangent bundle will be the product of the holonomy for the chiral primaries and the holonomy of the supercurrents. The latter contributes a factor of $\mathrm{SO}(4)$ associated to the $\mathrm{SO}(4)^{\mathrm{outer}}$ ambiguity of the supercurrents. So the moduli space is a $4n$ dimensional manifold whose holonomy K is reduced: $K \in \mathrm{SO}(4) \times \mathrm{SO}(n) \in \mathrm{SO}(4n)$. Such manifolds are constrained by Berger's classification. After a few more easy arguments [13] we conclude that the moduli space is a locally a homogeneous space of the form

$$\frac{\mathrm{SO}(4,n)}{\mathrm{SO}(4) \times \mathrm{SO}(n)} \tag{5.6}$$

This means that the local geometry of the moduli space is completely fixed by supersymmetry, and can be determined if we know the number of marginal operators which fixes n. In the case of $\mathrm{AdS}_3/\mathrm{CFT}_2$ we have $n = 5$ for $X = T^4$ and $n = 21$ for $X = K3$.[15]

Before we proceed, let us stress an important point. From each chiral primary ϕ_i with $(q, \overline{q}) = (\frac{1}{2}, \frac{1}{2})$ we get marginal operators which are singlets of $\mathrm{SO}(4)^R$

$$G^{-r}_{-1/2} \overline{G}^{-s}_{-1/2} \cdot \phi_i \tag{5.7}$$

[15] Notice that the moduli space is of the same form for all values of the central charge, so it seems to be independent of Q_1, Q_5. However we have not fixed the overall scale of the metric on the coset. This scale does depend on the central charge.

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where r, s can take any value in $\{+,-\}$ independently. We can also consider operators of the form

$$G^{+r}_{-1/2}\overline{G}^{+s}_{-1/2}\cdot\overline{\phi_i} \tag{5.8}$$

A very important property is that the tangent space of the moduli space is completely spanned by the operators of the form (5.7) alone. The same is true about the operators of the form (5.8). This can be roughly understood from the counting. The tangent space of the moduli space has real dimension $4n$. The set of operators of the form (5.8) has real dimension $8n$, but we have to impose a reality condition for the operator used to deform the theory so we are left with half of them which is equal to $4n$. It is not difficult to prove this property from the commutation relations of the $\mathcal{N}=\triangle,\triangle$ algebra.

Moreover, starting from a (cc) chiral primary ϕ of charge $(1/2,1/2)$ we can use the $\mathrm{SU}(2)^R_{\mathrm{right}}$ to rotate it to a (ca) primary ψ of charge $(1/2,-1/2)$. This will also lead to marginal operators of the form

$$G^{-r}_{-1/2}\overline{G}^{+s}_{-1/2}\cdot\psi_i \tag{5.9}$$

$$G^{+r}_{-1/2}\overline{G}^{-s}_{-1/2}\cdot\overline{\psi_i} \tag{5.10}$$

Again each of the two sets (5.7), (5.10) fully spans the tangent space. To summarize, from each chiral primary ϕ_i with $(q,\overline{q})=(\frac{1}{2},\frac{1}{2})$ we get 4 real marginal operators which are singlets of $\mathrm{SO}(4)^R$ and which transform under $\mathrm{SO}(4)^{\mathrm{outer}}$. These operators can be written in different ways (5.7), (5.8), (5.9), (5.10).

From each chiral primary ϕ_i with $(q,\overline{q})=(1,1)$ we get a single real operator with $(L_0,\overline{L}_0)=(2,2)$ which is a singlet of $\mathrm{SO}(4)^R\times\mathrm{SO}(4)^{\mathrm{outer}}$. These are the only irrelevant operators that exist which preserve global $\mathcal{N}=(4,4)$ supersymmetry but which break conformal invariance. We emphasize that this is a finite number of irrelevant operators.

6 The chiral ring of $\mathcal{N}=(4,4)$ theories

Finally, we are ready to consider the moduli dependence of the chiral ring in $\mathcal{N}=(4,4)$ superconformal field theories.

6.1 Curvature of the $\mathcal{N}=(4,4)$ algebra, the bosonic currents

We start with the curvature of the generators of the algebra. In principle their curvature can take values in the automorphism group $\mathrm{SO}(4)^R\times\mathrm{SO}(4)^{\mathrm{outer}}$ of the $\mathcal{N}=(4,4)$ algebra. To compute the curvature of the R-currents $J^i(z)$ we need the following 4-point function

$$\langle\mathcal{O}_\mu(x)\mathcal{O}_\nu(y)J^i(z)J^j(w)\rangle \tag{6.1}$$

where $\mathcal{O}_\mu,\mathcal{O}_\nu$ are marginal operators. As a function of z this 4-point function is holomorphic so it is completely determined by its singularity structure when $J^i(z)$ approaches the other insertions. We have the following OPEs

$$\begin{aligned} J^i(z)\mathcal{O}_\mu(x) &= \text{regular}\\ J^i(z)J^j(w) &= \frac{k}{2}\frac{\delta^{ij}}{(z-w)^2}+i\frac{\epsilon_{ijk}J^k(w)}{z-w}+\dots \end{aligned} \tag{6.2}$$

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The proof of the first OPE is based on the fact that in an $\mathcal{N} = (4,4)$ SCFT the marginal operators are descendants of chiral primaries, see appendix G for details.

Since the OPE of a current with a marginal operator is regular, the only contribution to the 4-point function is when the two currents come together. Then we have to use the second OPE in (6.2). The second term of that OPE involves $J^k(w)$ and is charged under $\mathrm{SO}(4)^R$, so its 3-point function with the neutral marginal operators is zero. So only the first term of the JJ OPE contributes and we find

$$\langle \mathcal{O}_\mu(x)\mathcal{O}_\nu(y)J^i(z)J^j(w)\rangle = \frac{k}{2}\frac{g_{\mu\nu}\delta^{ij}}{|x-y|^4(z-w)^2} \tag{6.3}$$

where $g_{\mu\nu}$ is defined by the two point function

$$\langle \mathcal{O}_\mu(x)\mathcal{O}_\nu(y)\rangle = \frac{g_{\mu\nu}}{|x-y|^4} \tag{6.4}$$

The 4-point function (6.3) is symmetric in $\mu \leftrightarrow \nu$, and has no singularities as $x, y \to z, w$ so the curvature of the R-currents, according to (3.29), is zero.

The conclusion is that there is no curvature for the $\mathrm{SO}(4)^R$ symmetry over the moduli space. From the AdS/CFT point of view this is according to our expectations. The R-symmetry of the CFT corresponds to the isometry group of the three-sphere in $AdS_3 \times S^3 \times K3$. Intuitively we expect that changing the moduli of the compactification should not induce a rotation of the S^3. The CFT analysis verifies this intuition.

6.2 The supercurrents

The supercurrents are charged under both the R-symmetry $\mathrm{SO}(4)^R$ and the outer automorphism $\mathrm{SO}(4)^{\mathrm{outer}}$. We found that the R-symmetry does not have curvature over the moduli space. However, as we will see the supercurrents mix among themselves by an $\mathrm{SO}(4)^{\mathrm{outer}}$ rotation. In principle this curvature can be computed by an analysis of 4-point functions of two supercurrents with two marginal operators, as in section 4.1. A faster way to derive the answer is the following. In sections 5.3, 5.4 we explained that the marginal operators are constructed by acting with supercurrents on chiral primaries of charge $\left(\frac{1}{2},\frac{1}{2}\right)$. If we call $\mathcal{G}^L, \mathcal{G}^R$ the bundles of left and right-moving supercurrents, $\mathcal{V}_{1/2,1/2}$ the bundle of chiral primaries of charge $\left(\frac{1}{2},\frac{1}{2}\right)$ and $\hat{\mathcal{O}}$ the bundle of marginal operators, then clearly $\hat{\mathcal{O}}$ is the tensor product of the other three bundles

$$\hat{\mathcal{O}} = \mathcal{G}^L \times \mathcal{G}^R \times \mathcal{V}_{1/2,1/2} \tag{6.5}$$

Moreover, $\hat{\mathcal{O}}$ is isomorphic to the tangent bundle $T\mathcal{M}_{\mathrm{CFT}}$ of the moduli space (5.6). The connection on $T\mathcal{M}_{\mathrm{CFT}}$ is described by the spin connection on the coset (5.6) which takes values in its isotropy group $\mathrm{SO}(4) \times \mathrm{SO}(n) = \mathrm{SU}(2)^L \times \mathrm{SU}(2)^R \times \mathrm{SO}(n)$. From the tensor product structure (6.5), it is clear that the connection of $\mathcal{G}^L$ is given by the $\mathrm{SU}(2)^L$ factor of the connection of the tangent bundle, the connection of $\mathcal{G}^R$ by $\mathrm{SU}(2)^R$ and that of $\mathcal{V}_{1/2,1/2}$ by the $\mathrm{SO}(n)$ factor. It should be easy to rederive this from a CFT computation of the 4-point functions, as in section 4.1. The main point is that the supercurrents have nonzero $\mathrm{SO}(4)^{\mathrm{outer}}$ curvature which is directly computable by the geometry of the coset (5.6) without any further input from the dynamics of the CFT.

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6.3 The 3-point functions are covariantly constant

We denote by ϕ_i the chiral primary fields of the $\mathcal{N}=(4,4)$ theory, that is, fields which are Virasoro primaries and satisfy $L_0 = J_0^3, \overline{L}_0 = \overline{J_0}^3$. Their OPE has the form

$$\phi_i(z)\phi_j(w) = C_{ij}^k \phi_k(w) + \dots \tag{6.6}$$

where ϕ_k is also chiral primary and C_{ij}^k are the structure constants of the chiral ring. We can also consider the 2- and 3-point functions related by

$$\begin{aligned} C_{ij\overline{k}} &= \langle \phi_i(0)\phi_j(1)\overline{\phi_k}(\infty)\rangle \\ g_{i\overline{j}} &= \langle \phi_i(0)\overline{\phi_j}(\infty)\rangle \\ C_{ij\overline{k}} &= C_{ij}^l g_{l\overline{k}} \end{aligned} \tag{6.7}$$

We want to compute how the chiral ring 3-point functions vary as we move on the moduli space. For this we need to compute the 4-point function

$$\langle \mathcal{O}(z)\phi_i(z_1)\phi_j(z_2)\overline{\phi_k}(z_3)\rangle \tag{6.8}$$

where $\mathcal{O}(z)$ is a marginal operator. We want to show that this 4-point function is zero.

What is important for the proof is that, as explained in section 5.4, for $\mathcal{N}=(4,4)$ theories any marginal operator can be written as the linear combination of descendants of antichiral primaries[16]

$$\mathcal{O}(z) = A_{rs}^{\overline{j}} G_{-1/2}^{+r} \overline{G}_{-1/2}^{+s} \cdot \overline{\phi_j}(z) \tag{6.9}$$

where $A_{rs}^{\overline{j}}$ are some appropriate constants.

Now we consider the Ward identity [31, 32]: on the sphere for a current $G^{+r}(w)$ of dimension 3/2. For any set of primary operators φ we have

$$\langle \oint \xi(w) G^{+r}(w)\varphi_{i_1}(z_1)\dots\varphi_{i_n}(z_n)\rangle = \sum_{i=1}^{n} \xi(z_i)\langle \varphi_{i_1}(z_1)\dots(G_{-1/2}^{r+}\cdot\varphi_i)(z_i)\dots\varphi_{i_n}(z_n)\rangle = 0 \tag{6.10}$$

where $\xi(w)$ is a globally defined holomorphic vector field of the form[17]

$$\xi(w) = aw + b \tag{6.11}$$

where a, b are arbitrary complex numbers. Using this Ward identity we have

$$\xi(z)\langle \mathcal{O}(z)\phi_i(z_1)\phi_j(z_2)\overline{\phi_k}(z_3)\rangle + \xi(z_3)\langle \left(A_{rs}^{\overline{j}}\overline{G}_{-1/2}^{+s}\cdot\overline{\phi_j}(z)\right)\phi_i(z_1)\phi_j(z_2))(G_{-1/2}^{+r}\cdot\overline{\phi_k})(z_3)\rangle = 0 \tag{6.12}$$

[16] Of course this is not true in $(2,2)$ theories, which is why in those theories we have $\nabla_{\overline{m}} C_{ijk} = 0$, but in general $\nabla_m C_{ijk} \neq 0$.

[17] We do not consider conformal killing vector fields of the form $\xi(w) \sim w^2$ for the following reason: since $G^{+r}(w)$ has dimension 3/2 the correlator $\langle G^{+r}(w)\varphi_{i_1}(z_1)\dots\varphi_{i_n}(z_n)\rangle$ falls-off like $\frac{1}{w^3}$ as $w \to \infty$. So if we do not want to have a contribution from infinity $\xi(w)$ can be at most linear in w.

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where we used $G^{r+}_{-1/2} \cdot \phi_i = G^{r+}_{-1/2} \cdot \phi_j = 0$ since these fields are chiral primaries. Now if we choose the vector field $\xi(w)$ in (6.11) is such a way that $\xi(z) = 1$ and $\xi(z_3) = 0$ we immediately get

$$\langle \mathcal{O}(z)\phi_i(z_1)\phi_j(z_2)\overline{\phi_k}(z_3)\rangle = 0 \tag{6.13}$$

Since the 4-point function vanishes for all marginal directions, following the definition of the covariant derivative (3.18) we find that there is no need for any subtraction and the covariant derivative of the 3-point function is zero. This means that the chiral ring is covariantly constant

$$\nabla C^k_{ij} = 0, \qquad \nabla C_{ij\overline{k}} = 0 \tag{6.14}$$

where we obviously also have $\nabla g_{i\overline{j}} = 0$ since the connection is compatible with the metric (3.24). This shows the non-renormalization of 3-point functions of chiral primaries.[18] The result is valid for any $\mathcal{N} = (4,4)$ theory, in particular in $\mathrm{AdS}_3/\mathrm{CFT}_2$ it is true not only in the large N limit but even for finite values of $N = Q_1Q_5$.

6.4 Non-renormalization of extremal correlators

More generally the same argument can be used to show that extremal correlators of the form

$$\langle \phi_{i_1}(z_1)\dots\phi_{i_n}(z_n)\overline{\phi_j}(y)\rangle \tag{6.15}$$

are also not renormalized.[19] For this we need the $n+2$-point function

$$\langle \mathcal{O}(z)\phi_{i_1}(z_1)\dots\phi_{i_n}(z_n)\overline{\phi_j}(y)\rangle \tag{6.16}$$

where $\mathcal{O}(z)$ is a marginal operator written as the descendant of an antichiral primary (6.9). We then follow the same steps as before. We use the Ward identity for the supercurrent G^{+r} by appropriately choosing $\xi(w)$ to have the value one at z and zero at y. All the fields at z_i do not contribute since they are chiral primaries and they are annihilated by $G^{+r}_{-1/2}$. So the $n+2$-point function (6.16) vanishes

$$\langle \mathcal{O}(z)\phi_{i_1}(z_1)\dots\phi_{i_n}(z_n)\overline{\phi_j}(w)\rangle = 0. \tag{6.17}$$

This means that the extremal correlator (6.15) is covariantly constant over the moduli space and receives no renormalizations.

Of course the same argument cannot be applied if we have at the same time two or more chiral fields and two or more antichiral fields since then we cannot choose ξ appropriately to cancel all contributions.

[18] It might be possible to argue that certain correlators of short multiplets in $\mathcal{N} = (4,4)$ SCFTs respect an $\mathrm{SO}(4)^{\mathrm{outer}}$ selection rule, even though the latter is not a proper symmetry of the full theory, in analogy with the "bonus" $\mathrm{U}(1)_Y$ symmetry in $\mathcal{N} = 4$ [9, 10]. From this selection rule the non-renormalization of 3-point functions would follow.

[19] We would like to thank R. Gopakumar and S. Minwalla for bringing this point to our attention.

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6.5 The curvature of chiral primaries

The chiral ring is a multiplication between chiral primaries. The chiral primaries themselves are sections of bundles $\mathcal{V}_q$ with nontrivial connections. We showed that the multiplication between these bundles

$$C_{ij}^k : \mathcal{V}_p \otimes \mathcal{V}_q \to \mathcal{V}_{p+q} \tag{6.18}$$

is covariantly constant. However this does not mean that the bundles are flat. In this section we want to compute the curvature of the bundles of chiral primaries for $\mathcal{N} = (4,4)$ theories.

We proceed by using the fact that the $\mathcal{N} = (4,4)$ algebra has many inequivalent $\mathcal{N} = (2,2)$ subalgebras. If we consider two marginal operators which are descendants of the (cc) and (aa) ring of a specific $\mathcal{N} = (2,2)$ subalgebra, we can compute the curvature along these two directions by using the results of our analysis in section 4. Then by varying the chosen $\mathcal{N} = (2,2)$ subalgebra we can effectively scan all (pairs of) directions on the moduli space and thus compute the curvature in all directions.

As we explained in section 5.1, to pick an $\mathcal{N} = (2,2)$ subalgebra of the $\mathcal{N} = (4,4)$ theory, we first need to pick Cartan elements of the $\mathrm{SO}(4)^R$. Let us take them to be $(J^3, \overline{J}^3)$. Then we have the $\mathrm{SO}(4)^{\text{outer}}$ ambiguity in choosing the supercurrents. Following [33, 34] we can define

$$\begin{aligned} \widehat{G^+}(u) &= \ u_1 G^{++} + u_2 G^{+-} \\ \widehat{G^-}(u) &= u_1^* G^{--} + u_2^* G^{-+} \end{aligned} \tag{6.19}$$

for any complex numbers u_1, u_2 satisfying $|u_1|^2 + |u_2|^2 = 1$. Then the currents $T(z), \widehat{G^\pm}(z), J^3(z)$ satisfy the standard $\mathcal{N} = 2$ superconformal algebra OPEs. We can do the same on the right-moving sector where we also have to choose complex numbers $\overline{u}_1, \overline{u}_2$ satisfying $|\overline{u}_1|^2 + |\overline{u}_2|^2 = 1$. Let us combine all these complex numbers in the symbol $\mathcal{U} = (u_1, u_2, \overline{u}_1, \overline{u}_2)$. Now consider the marginal operators

$$\begin{aligned} \mathcal{O}_{(\mathcal{U},i)} &= \frac{1}{2}\widehat{G^-}(u)\widehat{\overline{G}^-}(\overline{u}) \cdot \phi_i \\ \overline{\mathcal{O}_{(\mathcal{U},j)}} &= \frac{1}{2}\widehat{G^+}(u)\widehat{\overline{G}^+}(\overline{u}) \cdot \overline{\phi_j} \end{aligned} \tag{6.20}$$

where ϕ_i are (cc) fields and $\overline{\phi_j}$ are (aa) fields. The curvature along any pair of marginal operators of this form can be computed from the tt^* equations and we have

$$\begin{aligned} &[\nabla_{(\mathcal{U},i)}, \nabla_{(\mathcal{U},j)}] = [\nabla_{(\mathcal{U},\bar{i})}, \nabla_{(\mathcal{U},\bar{j})}] = 0 \\ &[\nabla_{(\mathcal{U},i)}, \nabla_{(\mathcal{U},\bar{j})}] = g_{i\bar{j}} g_{k\bar{l}} \left(1 - \frac{3}{c}(q + \overline{q})\right) - [C_i, \overline{C}_{\bar{j}}] \end{aligned} \tag{6.21}$$

for all possible $\mathcal{U}$'s and where $\nabla_{(\mathcal{U},i)}$ denotes the covariant derivative with respect to the marginal operator $\mathcal{O}_{(\mathcal{U},i)}$. By varying $\mathcal{U}$ these equations give us the curvature in all possible directions of the moduli space. In other words, if we want to compute the curvature of the bundle of chiral primaries along two specific tangent vectors on the moduli space, then there is enough freedom to rewrite the curvature operator in those direction as a linear

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combination of the curvature along pairs of vectors where for each pair the factor $\mathcal{U}$ is the same and we can use (6.21). Crucial here is the observation that for any $\mathcal{U}_1$ and $\mathcal{U}_2$, we can always rewrite $\mathcal{O}_{(\mathcal{U}_2,i)}$ as a linear combination of $\mathcal{O}_{(\mathcal{U}_1,i)}$ and $\overline{\mathcal{O}_{(\mathcal{U}_1,i)}}$.

6.6 Real structure of the chiral ring

In $\mathcal{N}=(4,4)$ theories it is more convenient to use a real basis for the chiral ring. Consider the (cc) primaries ϕ_i of charge $(q,\overline{q})$. We can transform them into (aa) fields in two ways. First, we can take the hermitian conjugate

$$\phi \to \overline{\phi_i} \equiv \phi_i^\dagger \tag{6.22}$$

Second, we can rotate them using the $\mathrm{SU}(2)^R_{\text{left}} \times \mathrm{SU}(2)^R_{\text{right}}$

$$\phi_i \to \widetilde{\phi}_i = \frac{1}{T_{q,\overline{q}}}(J^{--})^{2q}(\overline{J}^{--})^{2\overline{q}} \cdot \phi_i \tag{6.23}$$

where $T_{q,\overline{q}}$ is a real normalization factor chosen in such a way that the norm of $|\phi_i\rangle$ equals the norm of $|\widetilde{\phi}_i\rangle$. These two procedures generate the same set of (aa) fields, so there must be a matrix M relating the two

$$\overline{\phi_i} = M_{\bar{i}}^{j}\widetilde{\phi}_j \tag{6.24}$$

where M must satisfy

$$MM^* = I. \tag{6.25}$$

It is convenient to pick a basis $\phi_I, I=1,\ldots,n$ for the (cc) fields in which $M_{\bar{i}}^{j} = \delta_{\bar{i}}^{j}$. Then

$$(\phi_I)^\dagger = \frac{1}{T_{q,\overline{q}}}(J^{--})^{2q}(\overline{J}^{--})^{2\overline{q}} \cdot \phi_I, \tag{6.26}$$

In this basis the metric G_{IJ} becomes real, and by a second (real) change of basis we can take it to be δ_{IJ}

$$\langle \phi_I(z)\overline{\phi_J}(w)\rangle = \frac{\delta_{IJ}}{|z-w|^4} \tag{6.27}$$

Moreover, in this basis the chiral ring coefficients are also real

$$\begin{aligned} &\phi_I(z)\phi_J(w) = C_{IJ}^K\phi_K(w) + \ldots \\ &\left(C_{IJ}^K\right)^* = C_{IJ}^K \end{aligned} \tag{6.28}$$

Notice that since the action of J^{--} does not change under parallel transport (since we computed that the curvature of the currents J^i is zero), and also the action of the $\dagger$ on operators is unambiguously defined, it means that the choice of a real basis is invariant under parallel transport. The bundles $\mathcal{V}_q$ of chiral primaries are actually *real* vector bundles in the case of $\mathcal{N}=(4,4)$ theories.[20]

[20] Notice that when we say "real basis" we do not mean that the operators ϕ_I satisfy $\phi_I = \phi_I^\dagger$, which is impossible for operators of definite nonzero R-charge. Instead what we mean is that in this basis the inner product and the chiral ring coefficients between these operators become real. The actual operators remain "complex", or geometrically the (p,q) differential forms in the target space corresponding to the chiral primaries are still complex forms.

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6.7 Final expression for the curvature

Now let us consider the tangent space of the moduli space (5.6). The holonomy group is $\mathrm{SO}(4) \times \mathrm{SO}(n)$, so it is convenient to pick a vielbein basis where the tangent vectors are decomposed as $X^\mu = X^{a,I}$, where a transforms under $\mathrm{SO}(4)$ and I under $\mathrm{SO}(n)$. These tangent vectors correspond to marginal operators, which can be written as descendants of chiral primaries of charge $(1/2, 1/2)$ as in (5.7). The index I is associated to the chiral primary ϕ_I of charge $(1/2, 1/2)$ in the real basis described above, while the index a is related to the combination of the supercurrents $G^{-\pm}$ and $\overline{G}^{-\pm}$ that we act with on the chiral primary to get the marginal operator. From (6.21) it is easy to see that curvature of the bundle of chiral primaries in a real basis has the form

$$(R_{\mu\nu})^N_M = \delta_{ab}\delta_{IJ}\delta^N_M \left(1 - \frac{3}{c}(q+\overline{q})\right) - \delta_{ab}\left(C^K_{IM}\delta_{KL}C^L_{JP}\delta^{PN} - \delta_{MP}C^P_{JK}\delta^{KL}C^N_{IL}\right) \quad (6.29)$$

where the indices $\mu = (a, I), \nu = (b, J)$ denote two tangent directions μ, ν decomposed into their $\mathrm{SO}(4) \times \mathrm{SO}(n)$ factors.[21] Notice that from R-charge conservation, if the fields M, N have charge $(q, \overline{q})$ then the sum over K, L in the second term of (6.29) runs over fields with charge $(q + 1/2, \overline{q} + 1/2)$, while in the third term over fields with charge $(q - 1/2, \overline{q} - 1/2)$. So the curvature of the chiral primaries of given charge is determined by the chiral ring coefficients of them with those which are one unit of charge higher and one unit of charge lower. The curvature can be written as

$$R_{\mu\nu} = \delta_{ab}\delta_{IJ}\left(1 - \frac{3}{c}(q+\overline{q})\right) - \delta_{ab}(C_I C^T_J - C^T_J C_I) \quad (6.30)$$

Where we have not shown the matrix indices M, N on the curvature operator and it is always implied that $\mu = (a, I), \nu = (b, J)$. We will continue to use this condensed notation in the rest of this section and hope it will not cause any confusion.

We remind that for the 3-point functions we have

$$\nabla C^K_{IJ} = 0. \quad (6.31)$$

[21] Notice that the first term in the curvature is symmetric in μ, ν, which seems unacceptable for a curvature operator. However this term should precisely cancel the symmetric part of the second term, so that the total expression for the curvature is actually antisymmetric. Some simple examples of these cancellations were seen in section 4.4. While the antisymmetry of the curvature operator is guaranteed from general principles (since the connection is compatible with the Zamolodchikov metric) it is not manifest in the form (6.29). A small check is to consider the trace of the curvature, that is the case $I = J$. Then we can see from the target space point of view that the term $(C_I C^T_J - C^T_J C_I)$ is proportional to the commutator $[L, \Lambda]$ where the operator L is multiplication with the Kähler form and Λ the adjoint operator. From standard arguments this is a commutator of the Lefschetz SU(2) algebra where $J^+ = L, J^- = \Lambda, J^3 = (q + \overline{q} - \dim(M))/2$ where $\dim = c/3$ is the complex dimension of the target space and the operators are acting on $(q, \overline{q})$ forms. Thus we have $[L, \Lambda] = (q + \overline{q} - \dim(M))/2$. Then the trace of the second term in (6.29) is proportional to the first term up to a factor of $c/3$. This factor is explained in the following way: we have normalized the ϕ_I, ϕ_J operators so that their 2-point function is (6.27). On the other hand the two point function of the Kähler form should be proportional to $c/3$, as can be seen from the current correlator $\langle JJ \rangle \sim c/3$. Taking this factor into account we find that the trace of 6.29 exactly cancels.

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6.8 Geometry of the bundles

Since all the quantities appearing on the right hand side of (6.29) are covariantly constant, it means that the curvature operator is also covariantly constant

$$\nabla R_{\mu\nu} = 0. \tag{6.32}$$

Bundles of covariantly constant curvature over homogeneous spaces, such as the moduli space (5.6), are called *homogeneous bundles*. It is a mathematical theorem [35] that the connection on homogeneous bundles is completely determined by the connection on the tangent bundle of the underlying base space, in our case (5.6). Each homogeneous bundle is characterized by a representation $\mathcal{R}$ of the holonomy group $\mathrm{SO}(4) \times \mathrm{SO}(n)$ and the connection on it is the same as that of the tangent bundle but in the representation $\mathcal{R}$.[22] Actually, from the expression (6.30) for the curvature we see from the factor δ_{ab} that the SO(4) representation is always the trivial one.

So finally, the geometry of the bundle $\mathcal{V}_q$ of chiral primaries of charge q is completely characterized by a (possibly reducible) representation $\mathcal{R}$ of $\mathrm{SO}(n)$. To determine the representation we have to consider the $\mathrm{SO}(n)$ part of the curvature operator

$$(C_I C_J^T - C_J^T C_I)_M^N \tag{6.33}$$

This has to decompose into representations $\mathcal{R}_k$ of $\mathrm{SO}(n)$. Then the bundle of chiral primaries of charge q is the direct sum of homogeneous bundles corresponding to these representations

$$\mathcal{V}_q = \sum_k \oplus \mathcal{V}_{\mathcal{R}_k} \tag{6.34}$$

The geometry of each of $\mathcal{V}_{\mathcal{R}_k}$ is completely fixed by the geometry of the coset

$$\frac{\mathrm{SO}(4,n)}{\mathrm{SO}(4) \times \mathrm{SO}(n)} \tag{6.35}$$

and some basic group theory which is completely independent of the dynamics of the CFT. For example, the chiral primaries of charge $(1/2, 1/2)$ always transform in the vector representation of $\mathrm{SO}(n)$ and the corresponding bundle $\mathcal{V}_{(1/2,1/2)}$ has curvature of the form

$$R_{\mu\nu} = -f \delta_{ab} \Sigma_{IJ} \tag{6.36}$$

where $(\Sigma_{IJ})_M^N$ are matrices in the vector representation of the $\mathrm{SO}(n)$ algebra, that is they satisfy

$$[\Sigma_{IJ}, \Sigma_{KL}] = \delta_{JK}\Sigma_{IL} + \delta_{IL}\Sigma_{JK} - \delta_{JL}\Sigma_{IK} - \delta_{IK}\Sigma_{JL} \tag{6.37}$$

and f is a numerical constant which depends on the overall scale of the coset (6.35). In the case of the D1/D5 CFT f is inversely proportional to the central charge of the

[22] If L is the vector space that carries the representation $\mathcal{R}$ then the vector bundle is explicitly constructed as $(\mathrm{SO}(4,n) \times L)/(\mathrm{SO}(4) \times \mathrm{SO}(n))$.

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theory. Similarly for a bundle in the representation $\mathcal{R}$ we have matrices $\Sigma^{\mathcal{R}}_{IJ}$ of the SO(n) algebra (6.37) and the curvature operator $\mathcal{V}_{\mathcal{R}}$ is

$$R_{\mu\nu} = -f\delta_{ab}\Sigma^{\mathcal{R}}_{IJ}. \tag{6.38}$$

Notice that from the fact that the marginal operators are descendants of the $(1/2, 1/2)$ chiral primaries and using the curvature (6.36) for these fields and the corresponding curvature for the supercurrents we get the following expression for the curvature of the marginal operators

$$(R_{\mu\nu})^{\lambda}_{\kappa} = f\left((\sigma_{ab})^{d}_{c}(\delta_{IJ})^{L}_{K} - (\delta_{ab})^{d}_{c}(\Sigma_{IJ})^{L}_{K}\right) \tag{6.39}$$

where σ_{ab} is the vector representation of SO(4) and again we use the notation $\mu = (a, I), \nu = (b, J), \kappa = (c, K), \lambda = (d, L)$. It is easy to recognize that (6.39) is the curvature of the tangent bundle of the coset (6.35) in a vielbein basis and where f controls the overall size of the manifold.

In practice, if we can compute the curvature operator from the 3-point functions at one point of the moduli space then we can find the decomposition of chiral primaries into representations of SO(n) and fix the geometry of the bundles, at least in a neighborhood of the point. For example in $\mathrm{AdS}_3/\mathrm{CFT}_2$ such a point could correspond to the orbifold CFT.

6.9 Example: IIB on $K3$

Let us now explain how the previous arguments apply to the case of IIB on $AdS_3 \times S^3 \times K3$. This is the near horizon geometry of a bound state of Q_1 D1 and Q_5 D5 branes wrapped on $K3$. The boundary conformal field theory is believed to be described by a deformation of a supersymmetric sigma model whose target space is the orbifold $K3^N/S_N$, where $N = Q_1Q_5$. The moduli space is locally the coset

$$\frac{\mathrm{SO}(4,21)}{\mathrm{SO}(4)\times\mathrm{SO}(21)} \tag{6.40}$$

The holonomy of the tangent bundle of the moduli space is SO(4)×SO(21). As we explained before, the connection on the vector bundles of the chiral primaries will be associated to that of the tangent bundle and in particular to its SO(21) part. So each of these bundles will be characterized by a representation $\mathcal{R}$ of SO(21).

The chiral primary states of this theory can be conveniently encoded in the Poincaré polynomial[23]

$$P_{t,\bar{t}} = \mathrm{Tr}\left(t^{2J_0}\bar{t}^{2\bar{J}_0}\right) \tag{6.41}$$

where the trace is taken over the space of chiral primaries. The chiral primary states are related to harmonic forms in the target space and it can be shown that the Poincaré polynomial equals

$$P_{t,\bar{t}} = \sum_{p,q} h_{p,q} t^p\, \bar{t}^q \tag{6.42}$$

[23] This is in the $\mathcal{N} = 4$ conventions where the normalization of J_0 is half-integral.

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where $h^{p,q}$ are the Hodge numbers of the target space. The Hodge numbers of $K3$ are equal to

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{6.43}$$

Starting with the single $K3$, it is possible to compute the Hodge numbers of the resolution of $K3^N/S_N$ from the generating function [36]

$$\sum_{N\geq 0} Q^N P_{t,\bar{t}}(K3^N/S_N) = \prod_{m=1}^{\infty}\prod_{p,q}\left(1+(-1)^{p+q+1}Q^m t^{p+m-1}\bar{t}^{q+m-1}\right)^{(-1)^{p+q+1}h^{p,q}}. \tag{6.44}$$

From this expression we can compute the numbers of chiral primaries of given conformal dimension in the SCFT. As we mentioned before these numbers agree with the results obtained from supergravity.

Now let us look at the low lying chiral primaries and sketch how they fit into vector bundles over the moduli space. For large enough N the even Hodge numbers (all odd numbers are zero) of the Hilbert scheme $K3/S_N$ are

$$\begin{array}{c} \cdots\cdot \\ \cdots\cdot \\ 1\ 22\ 276\ 2278\ 276\ 22\ 1 \\ 1\quad 22\quad 254\quad 22\quad 1 \\ 1\quad 21\quad 1 \\ 1 \end{array} \tag{6.45}$$

Let us see how we can represent these chiral primaries in the orbifold CFT language [37, 38]. We introduce bosonic creation operators α^A_{-n}, where $n = 1, 2\ldots$ labels the level of the twisted sector and A runs over the Dolbeault cohomology classes of a single $K3$. For a given (p,q), there are $\dim H^{(p,q)}(K3)$ operators $\alpha^{(p,q)}_{-n}$. The general chiral primary can be written as

$$\prod_{i=1}^{M}\alpha^{A_i}_{-n_i}|0\rangle, \qquad \sum_i^M n_i = N. \tag{6.46}$$

The R-charge of this operator is

$$\left(J^3, \overline{J}^3\right) = \frac{1}{2}\left(N - M + \sum_i p_i, \quad N - M + \sum_i q_i\right). \tag{6.47}$$

There is only one operator of charge $(0,0)$ which we will denote by $|N\rangle$. It is given by the product $|N\rangle \equiv \prod_{i=1}^N \alpha^{0,0}_{-1}|0\rangle$; clearly, this is to be identified with the identity operator and of course there is no holonomy for it. We have a single operator of charge $(1,0)$, which may be represented as $\alpha^{2,0}_{-2}|N-1\rangle$. The operator with charge $(0,1)$ is similarly represented by $\alpha^{0,2}_{-2}|N-1\rangle$. They correspond to the R-symmetry currents $J^3, \overline{J}^3$. As we saw in section 6.1 the holonomy for these operators is also trivial.

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Now we consider the 21 operators of charge $(\frac{1}{2},\frac{1}{2})$. They are given by the following products of creation operators

$$20 \times \qquad \alpha_{-1}^{(1,1)}\,|N-1\rangle \quad \text{and} \quad 1 \times \qquad \alpha_{-2}^{(0,0)}\,|N-2\rangle\,. \tag{6.48}$$

From this we conclude that the operators of charge $(\frac{1}{2},\frac{1}{2})$ fall into the vector representation **21** of SO(21).[24] The connection of this bundle over the moduli space is the same as the SO(21) part of the tangent bundle of (6.40). Acting on a each of these states with one left-moving and one right-moving supercurrent gives the $4 \times 21 = 84$ marginal operators.

At higher conformal dimension, we have to distinguish between single-particle and multi-particle chiral primaries. A multi-particle field is given by the product of chiral primaries of lower charge, while a single-particle operator is a genuinely new chiral primary appearing at the given conformal dimension. For example if we look at the operators of charge $(1,1)$ we have 254 of them. We can have multi-particle states of the form $(1/2,1/2)\times(1/2,1/2)$ which are $(21\times 22)/2$ in number, or of the form $(1,0)\times(0,1)$, which is one state. So in total we have 232 multi-particle operators at this level and 22 single-particle ones.

The multi-particle states will obviously fall into tensor product representations of SO(21) determined by their decomposition into single-particle operators. The corresponding bundles are isomorphic to the tensor product of the bundles of their constituents. Hence the new information at each level is related to the the bundles of chiral primaries which are single-particle operators.

More generally, for m small enough compared to N, we have single-particle operators of total charge m only when the charge is of the form $\frac{1}{2}(m,m)$, $\frac{1}{2}(m+1,m-1)$ or $\frac{1}{2}(m-1,m+1)$. The single-particle operators[25] with charge $\frac{1}{2}(m,m)$ can be represented in the form

$$\alpha_{-m-1}^{(0,0)}|N-m-1\rangle, \quad \alpha_{-m}^{(1,1)}|N-m\rangle, \quad \alpha_{-m+1}^{(2,2)}|N-m+1\rangle. \tag{6.49}$$

so there are $1+20+1$ of them. Our natural guess is that they decompose as $\mathbf{21}+\mathbf{1}$ of the SO(21). In principle we could compute their 4-point function at the orbifold point and check whether this is indeed true. For $\frac{1}{2}(m+1,m-1)$ and $\frac{1}{2}(m-1,m+1)$ we have

$$\alpha_{-m}^{(2,0)}|N-m\rangle, \qquad \alpha_{-m}^{(0,2)}|N-m\rangle \tag{6.50}$$

respectively. They are obviously in the **1** of SO(21).

To summarize, we denote by $\mathcal{V}_{\mathbf{21}}$ the unique real vector bundle of rank 21 over the moduli space (6.40) whose connection is the same as the SO(21) part of the tangent bundle. The curvature of this bundle of the form (6.36). We denote by $\mathcal{V}_{\mathbf{1}}$ the trivial bundle of rank one and $\mathcal{V}_{\mathbf{multi}}$ the tensor product of vector bundles corresponding to the lower conformal dimensions. We have the following answer for the geometry of the vector bundle $\mathcal{V}_{p,q}$ of

[24] Another possibility is that they might be 21 singlets of SO(21). However we know that we get the marginal operators as descendants of these chiral primaries, and the marginal operators transform under SO(21), so this possibility is excluded.

[25] Notice that we are not careful about the precise linear combinations that gives us the single- vs multi-particle operators since we are only interested in their counting and not the actual operators.

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chiral primaries of charge (p,q) with $p,q>1$

$$\mathcal{V}_{p,q} = \begin{cases} \mathcal{V}_{\mathbf{multi}} \oplus \mathcal{V}_{\mathbf{21}} \oplus \mathcal{V}_{\mathbf{1}}, & \text{if} \quad p=q, \\ \mathcal{V}_{\mathbf{multi}} \oplus \mathcal{V}_{\mathbf{1}}, & \text{if} \quad p=q+1 \quad \text{or} \quad q=p+1, \\ \mathcal{V}_{\mathbf{multi}}, & \text{otherwise.} \end{cases} \tag{6.51}$$

Interestingly, if we look at the Fock space (6.46) then according to the previous discussion for each fixed $N>1$ it should carry a representation of SO(21). This representation is certainly not manifest. There is an obvious action of SO(20) which rotates the α^A_{-n} with A the 20 $(1,1)$-forms into each other and leaves the other α^A_{-n} fixed. The extra operators which extend SO(20) to SO(21) must be more complicated. If we also introduce the positive modes of the bosons with commutation relations

$$[\alpha^A_{-n}, \alpha^B_m] = m\delta_{n,m} \int_{K3} A \wedge B \tag{6.52}$$

then the SO(20) generators can be written as quadratic operators in the modes of the bosons. However, the extra SO(21) generators must be at least cubic. It would interesting to construct these generators explicitly and study their precise algebraic and geometrical meaning.[26]

Let us also notice that for given m, a single particle operator is a map from the cohomology of $K3$, $H^*(K3)$, to the cohomology of the symmetric product $H^*(\mathrm{Sym}^N(K3))$. In section 6.10 we will see how the chiral primaries can be identified with operators in the 4-dimensional gauge theory, which can also be interpreted as forms on the instanton moduli space.

6.10 Chiral primaries in 4d gauge theory

In the previous subsection, the chiral primaries of the 2d sigma model were considered. The target space of the sigma model is the moduli space $\mathcal{M}$ of instantons of 4d gauge theory on $K3$. Therefore, we might expect that the 2d chiral primaries have analogues in the gauge theory. Such a connection is potentially interesting since we might be able to learn more about the geometry of the chiral ring by the computation of gauge theory quantities like Donaldson polynomials. On the other hand, it might be useful for an analysis of the geometry of the chiral ring of the superconformal 4d gauge theory, see also section 9. The gauge theory can be obtained by wrapping the D5-brane system on $T^2 \otimes K3$, and considering the limit where the typical length scale of the T^2 is much smaller then the one of $K3$. In this way one ends up with $\mathcal{N}=4$ Yang-Mills theory on $K3$.

The correspondence between the 2d and 4d operators can be understood more precisely if we recall the representation of Donaldson polynomials in terms of the fields of $\mathcal{N}=2$ gauge theory in [39]. See also [16] for a discussion of Donaldson polynomials in the context of AdS/CFT. Similar to the interpretation of the single particle operators $\alpha^{A_i}_{-n}$ as differential forms on $\mathcal{M}$ in section 6.9, the Donaldson polynomials can be viewed as differential forms on $\mathcal{M}$. By a comparison of the infinitesimal deformations of an instanton solution and the

[26] As mentioned before, we believe that a class of correlators of short multiplets may respect an $\mathrm{SO}(4)^{\mathrm{outer}} \times \mathrm{SO}(21)$ selection rule as in [9, 10]. It would be interesting to clarify this point.

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supersymmetry transformations, ref. [39] assigns a form degree on $\mathcal{M}$ to the gauge theory fields. This degree corresponds to the charge under a U(1) subgroup of the R-symmetry group. The R-symmetry group of $\mathcal{N}=4$ Yang-Mills is SU(4), which can be decomposed as $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. The charge of a field under the last U(1) provides its form degree on $\mathcal{M}$. The field content of the theory is a gauge field A_μ, six scalars ϕ_i and fermions. Four of the scalars and A_μ have U(1)-charge 0, the two other scalars have charge $+2$ and -2, and the fermions have charges $+1$ or -1. The sixteen supersymmetry generators can be divided in two sets of eight, based on their U(1)-charge ± 1. The SU(2)-holonomy of $K3$ preserves half of the susy generators in both sets. We denote the preserved susy generators by $Q_\alpha^{\pm I}$, where $\pm$ denotes the U(1)-charge, α labels the space-time SU(2) which is preserved by the $K3$ holonomy, and $I=1,2$. A susy generator with charge $+1$ plays often a distinguished role, namely when it is taken as the generator of a topological symmetry after twisting of the theory.

This also distinguishes the scalar with U(1)-charge 2 (which we denote by σ). This scalar is namely annihilated by the susy generators with charge $+1$, because a field with charge $+3$ does not exist. These susy generators are the analogues of the operators $G^{+\pm}_{-1/2}$ and $\bar{G}^{+\pm}_{-1/2}$, which annihilate the states of the chiral-chiral ring. Among the operators which are the analogues of the states in the chiral-chiral ring are thus $W_0^m=\mathrm{Tr}\,(\sigma^m)$. These are not all the operators which are annihilated by Q_α^{+I}. As explained in [39], one can construct descendants W_k^m of W_0^m, such that $dW_k^m\sim\left\{Q_\alpha^{+I},W_{k+1}^m\right\}$. These forms are given by

$$W_0^m=\mathrm{Tr}\,(\sigma^m)\,,\qquad W_2^m=\mathrm{Tr}\left(\sigma^{m-1}\wedge F\right),\qquad W_4^m=\mathrm{Tr}\left(\sigma^{m-2}\wedge F\wedge F\right),\tag{6.53}$$

where we have ignored the fermions. Since K3 does not contain odd-dimensional cycles, only those descendants are given which are related to even forms. Since acting with Q_α^{+I} results in a total derivative, the following non-local operators are invariant under Q_α^{+I}:

$$\int_{\mathrm{A_i}} W_2^m,\qquad \text{and}\qquad \int_{\mathrm{K3}} W_4^m,\tag{6.54}$$

where the A_i form a basis of the 22 two-cycles of K3. Since F is a zero-form on $\mathcal{M}$ and σ a two-form, these operators are respectively $2m-2$, and $2m-4$ forms on $\mathcal{M}$.

We have now constructed the set of operators, which are dual to the operators $\alpha_{-n}^{A_i}$. e.g. the operators in eq. (6.48) together with the currents J^3 and $\overline{J}^3$ have total charge 1, and are thus two-forms on $\mathcal{M}$. These operators correspond to integrated descendants of $\mathrm{Tr}(\sigma^2)$ and $\mathrm{Tr}(\sigma^3)$. They are explicitly given by

$$22\times\int_{\mathrm{A_i}} W_2^2,\quad\text{and}\quad 1\times\int_{\mathrm{K3}} W_4^3.\tag{6.55}$$

These are therefore the counter parts of the 23 chiral primaries with total charge 1 in section 6.9. It is conceivable that the two-cycles A_i, whose Poincaré dual is a $(1,1)$-form, correspond to the 20 operators in (6.48). We can easily go further and include the chiral primaries with larger charges. The single particle operators with total charge m in section 6.9 correspond to $2m$-forms on $\mathcal{M}$. These $2m$-forms on $\mathcal{M}$ correspond to the

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appropriate descendants of $\mathrm{Tr}(\sigma^m)$, $\mathrm{Tr}(\sigma^{m+1})$ and $\mathrm{Tr}(\sigma^{m+2})$, namely

$$1\times \quad W_0^m, \qquad 22\times \quad \int_{\mathrm{A_i}} W_2^{m+1}, \qquad \text{and} \quad 1\times \quad \int_{\mathrm{K3}} W_4^{m+2}. \tag{6.56}$$

Thus we have shown above that the chiral primaries of 2d CFT can be identified with operators in $\mathcal{N}=4$ Yang-Mills. The marginal operators in section 6.9 are obtained by acting with the operators $G^{-\pm}_{-1/2}$ and $\bar{G}^{-\pm}_{-1/2}$. These operators correspond in the gauge theory to the generators Q^{-I}_{α} with U(1)-charge -1. As mentioned before, we can also identify the gauge theory chiral primaries in terms of $\alpha^{\mathrm{A}_i}_{-1}$. For example, the operator $\mathrm{Tr}(\sigma^2)$, mentioned in section 9 as the gauge theory chiral primary, which has as descendant a marginal operator, corresponds to $\alpha^{(2,2)}_{-1}$.

7 Attractor mechanism and RG-flow

One of our original motivations for studying the moduli dependence of the chiral ring in $\mathcal{N}=(4,4)$ theories, was its possible relevance for the analysis of the connection between the attractor flow in supergravity and RG-flow in the dual field theory. The attractor mechanism is usually studied in the case of 4-dimensional extremal black holes, but more generally it also appears for extremal branes of other dimensionalities. The attractor mechanism is a consequence of the extremality of the brane and not of supersymmetry [40–42], however it is technically easier to study in the supersymmetric case. To keep our discussion simple, we will only consider the cases of spherically symmetric flows and will ignore all subtleties related to multiple attractor points, walls of marginal stability and split-flows. Obviously it would be extremely interesting to understand such phenomena from the RG-flow point of view but this is beyond the scope of our simple analysis.

7.1 The attractor mechanism

Consider a supergravity theory in D dimensions with a moduli space $\mathcal{M}_{\mathrm{sugra}}$ in which the massless scalar fields take values. We pick coordinates z on $\mathcal{M}_{\mathrm{sugra}}$. The metric on the moduli space is $g_{ab}(z)$. We assume that the theory admits BPS p-brane solutions, charged under $(p+2)$-form field strengths. The charge Γ of these branes takes values in a lattice Λ. A very useful quantity is the spacetime central charge of the brane

$$Z(\Gamma, z) \tag{7.1}$$

which is determined by the supersymmetry algebra[27] and is a function of the charge vector Γ and the position on the moduli space z. If we call z^∞ the values of the moduli at infinity, then the ADM mass/tension of the black brane in D-dimensional Planck units is equal to

$$M_{\mathrm{ADM}} = |Z(\Gamma, z^\infty)| \tag{7.2}$$

In the supergravity solution the moduli z evolve radially reaching constant values z^* near the horizon. The value z^* depends only on the charge Γ of the brane and not on the values

[27] For simplicity we assume that there is only one complex central charge.

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of the moduli at infinity z^∞. This is the attractor mechanism. The condition for z^* to be an attractor point is that is minimizes the central charge Z

$$\frac{\partial |Z|}{\partial z^i}|_{z=z^*} = 0 \tag{7.3}$$

For every charge vector $\Gamma \in \Lambda$, there is a submanifold of solutions of (7.3)

$$\mathcal{M}^{*,\Gamma}_{\text{sugra}} \in \mathcal{M}_{\text{sugra}} \tag{7.4}$$

of attractor points, that we call the attractor submanifold for the charge vector Γ.[28] The radial evolution of the moduli from their value z^∞ at infinity to z^* at the horizon is governed by the attractor flow.

For example, for a spherically symmetric 4d black hole in $\mathcal{N} = 2$ supergravity we have the ansatz

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}\left(dr^2 + r^2 d\Omega_2^2\right) \tag{7.5}$$

For supersymmetric solutions we can write first order flow equations for $U(r), z^a(r)$. It is more convenient to work with the coordinate $\tau = 1/r$. This leads to the following flow equations

$$\begin{aligned} \dot{U} &= -e^U |Z| \\ \dot{z}^a &= -2e^{2U} g^{a\bar{b}} \partial_{\bar{b}} |Z| \end{aligned} \tag{7.6}$$

Similar relations hold for black branes of higher dimensionality.

7.2 Relation to RG-flow

The attractor black holes discussed in the previous section can be realized in string theory by bound states of D-branes. In this description the D-branes are placed in a flat background space, where the values of the scalar moduli are equal to their asymptotic values z^∞. The supergravity solution arises after backreaction and then we see the attractor mechanism in the radial evolution of the moduli. We want to understand what is the meaning of the attractor mechanism in the original D-brane picture.

The open string excitations on the worldvolume of the D-branes can be described in an appropriate regime by an effective quantum field theory. The background values z^∞ of the closed string moduli enter the worldvolume theory in the form of coupling constants. We will call the set of parameters of the effective field theory on the branes $\mathcal{M}_{\text{QFT}}$, which we will (loosely) identify with $\mathcal{M}_{\text{sugra}}$. The supergravity description of the same system has an AdS throat in the near horizon region, which indicates that the worldvolume theory flows to a conformal field theory in the IR. Moreover, in the near horizon region the moduli reach their attractor values z^*.

[28] Usually we speak of attractor points and not submanifolds, however even in the familiar case of black holes in 4d $\mathcal{N} = 2$ theories, the vector multiplets are fixed by the attractor mechanism to isolated points, while the hypermultiplets can take any value. In this case $\mathcal{M}_{\text{sugra}} = M_{\text{vector}} \times M_{\text{hyper}}$ and the attractor submanifold will be $\mathcal{M}^{*,\Gamma}_{\text{sugra}} = \{p\} \times M_{\text{hyper}}$, where $\{p\} \in M_{\text{vector}}$ is the attractor point for the charge Γ.

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This suggests that the IR fixed point of the worldvolume theory only knows about the attractor values of the moduli, hence the moduli space $\mathcal{M}_{\rm CFT}$ of the conformal field theory has to be identified with the attractor submanifold $\mathcal{M}^{*,\Gamma}_{\rm sugra}$ in supergravity. In other words if we flow to the IR, the number of parameters of the worldvolume theory is generally reduced leaving us with $\mathcal{M}_{\rm CFT} \subset \mathcal{M}_{\rm QFT}$. It is reasonable to assume that the way in which the UV coupling constants on the D-brane theory transform into the effective IR ones is by renormalization group flow. In this sense the D-brane theory sees the attractor mechanism as RG-flow on its worldvolume. Then it is natural to expect that the attractor flow equations (7.6) will play the role of RG-flow equations in the space of effective coupling constants of the D-brane theory.

The RG-flow of the worldvolume theory is governed by the β functions, which describe the flow of the coupling constants as a function of the energy scale. More precisely the β functions correspond to a vector field on the space of parameters of the theory $\mathcal{M}_{\rm QFT}$. The flow lines of this field give RG-flow orbits which, for the class of the theories we are considering, approach conformal fixed points at low energies where $\beta = 0$. The set of these points constitute the moduli space $\mathcal{M}_{\rm CFT}$ of conformal theories inside the bigger space $\mathcal{M}_{\rm QFT}$ of effective quantum field theories. Similarly the attractor flow equations (7.6) describe the radial flow of the moduli in gravity from $\mathcal{M}_{\rm sugra}$ to the submanifold $\mathcal{M}^{*,\Gamma}_{\rm sugra}$. The two pictures are consistent if we accept the usual AdS/CFT intuition that the radial direction corresponds to the energy scale. The statement that more than one value of the moduli at infinity flow to the same value near the horizon is related to the fact that more than one UV quantum field theories can flow to the same IR fixed point.

It would certainly be very interesting to understand this connection in more detail, however making this intuitive picture more precise is not straightforward. Besides the fact that the worldvolume theory is generally strongly coupled, there is an important conceptual difficulty, that away from the conformal point in the IR, i.e. away from the strict $\alpha' \to 0$ limit, the theory on the branes is not decoupled from the closed string modes in the bulk.

While the absence of a decoupling limit may be a serious obstacle for a precise formulation of the attractor flow/RG-flow relation it should be possible to work in a perturbative expansion around the conformal point, i.e. to first order away from the $\alpha' \to 0$ limit. There it should be possible to study the relation between attractor flow and RG-flow reliably. In the rest of this section we will only consider the first order flow and leave the more difficult study of finite flows for future work.

Our goal is to start from the conformal point and consider a first-order perturbation towards the UV. At the conformal point we have the AdS/CFT duality between the AdS factor of the near horizon geometry and the conformal IR fixed point of the D-brane theory. To see the attractor mechanism we have to flow from the near horizon geometry towards asymptotic infinity. In the boundary theory this means that we have to study the IR conformal field theory perturbed by irrelevant operators (see [43] for a similar discussion in the case of $\mathrm{AdS}_5/\mathrm{CFT}_4$). Perturbing a field theory by irrelevant operators is dangerous since it drastically modifies its UV behavior. However, since we are only interested in the first order flow away from the fixed point we will treat the conformal field theory perturbed by irrelevant operators as an effective field theory and study RG-flow in the Wilsonian sense, even though we do not have a UV completion of the theory.

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In supergravity the entire attractor flow solution preserves the same amount of supersymmetry and spherical symmetry as the near horizon geometry[29] so on the boundary theory we will only consider perturbations by irrelevant operators which do not break the supersymmetry and R-symmetry of the CFT but only conformal invariance. As we will see in our toy model, this constrains the number of allowed irrelevant operators to a finite set.

Now we would like to make more precise the statement that the attractor flow and RG-flow agree to first order away from the fixed point. As we can see in figure 1 this means that the structure of the flow on the two sides should be the same in a neighborhood of the fixed submanifolds $\mathcal{M}_{\rm CFT}, \mathcal{M}^{*,\Gamma}_{\rm sugra}$. The "zeroth-order" matching of the two sides relates the geometry of the fixed submanifolds. This is a consequence of the AdS/CFT correspondence between the near horizon geometry of the extremal brane and the conformal field theory in the IR of the D-brane theory. So, at least locally, we must have[30]

$$\mathcal{M}^{*,\Gamma}_{\rm sugra} = \mathcal{M}_{\rm CFT} \tag{7.7}$$

This is statement about the dimensionality as well as the geometry of the two manifolds. The metric on $\mathcal{M}^{*,\Gamma}_{\rm sugra}$ is fixed by the metric g_{ab} on the moduli space $\mathcal{M}_{\rm sugra}$, but see also footnote (4). The metric on $\mathcal{M}_{\rm CFT}$ is determined by the Zamolodchikov metric of marginal operators in the CFT, which correspond to tangent vectors on $\mathcal{M}_{\rm CFT}$.

The next step is to consider the first order flow towards the UV. For the two sides to be in agreement, the number of allowed irrelevant operators must be the same as the codimension of the attractor submanifold inside the full moduli space of supergravity-which is equal to the number of fixed moduli. Also the conformal dimension of the irrelevant operators must be related to the mass of the fixed moduli in the near horizon geometry by the standard mass/conformal dimension relation in AdS/CFT.

Moreover, the identification between the attractor flow and RG-flow suggests that there should be a relation between the two parameter spaces, not only on the fixed submanifolds but also away from them, at least to first order. A way to state this more precisely is that the normal bundle $\mathcal{N}_{\rm sugra}$ of the attractor submanifold $\mathcal{M}^{*,\Gamma}_{\rm sugra}$ inside $\mathcal{M}_{\rm sugra}$ should have the same structure as the normal bundle $\mathcal{N}_{\rm CFT}$ of $\mathcal{M}_{\rm CFT}$ in $\mathcal{M}_{\rm QFT}$

$$\mathcal{N}_{\rm sugra} = \mathcal{N}_{\rm CFT} \tag{7.8}$$

The geometry of the bundle $\mathcal{N}_{\rm CFT}$ encodes how the CFT can be perturbed by irrelevant operators (which preserve certain symmetries). Its geometry is characterized by the Zamolodchikov metric and the connection for the irrelevant operators in the CFT. Notice that the identification (7.8) of the normal bundles requires not only a matching of their ranks, which is guaranteed if the number of irrelevant operators is the same as the number

[29] Except for the extra supercharges that we get in the AdS region which are dual to the superconformal generators in the CFT.

[30] This has been demonstrated in the case of $\mathrm{AdS}_3/\mathrm{CFT}_2$ with 16 supercharges. It would be interesting to prove the same statement for the $(0,4)$ MSW theory or even for 4d black holes. A naive approach would suggest that the moduli space of of the "superconformal quantum mechanics" on the D-branes should be related to the attractor submanifold, which in this case coincides with the hypermultiplet moduli space. We hope to address this question in the future.

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of fixed moduli, but also a matching of the connections on the two bundles. The connection on $\mathcal{N}_{\rm sugra}$ is easily computable if we know how $\mathcal{M}^{*,\Gamma}_{\rm sugra}$ is embedded in $\mathcal{M}_{\rm sugra}$, while the connection on $\mathcal{N}_{\rm CFT}$ in the CFT equals the connection for the irrelevant operators over the moduli space as was explained in section 3.

These three conditions, identification of moduli spaces (7.7), of number/dimension of irrelevant operators to number/mass of fixed moduli, and identification of the normal bundles (7.8) is enough to guarantee the identification of attractor flow to RG-flow to first order away from the conformal fixed point. As we see all these quantities can be computed within the CFT, so unless we want to go to higher orders in perturbation theory, we do not have to worry about the UV completion of the theory and issues related to the decoupling of closed string modes.[31]

7.3 The D1/D5 system

The simplest system where we could try to check the attractor flow/RG-flow connnection is the D1/D5 bound state.[32] We start with IIB compactified on $K3$. This leads to chiral $(2,0)$ supergravity in 6 dimensions [44], whose moduli space is

$$\mathcal{M}_{\rm sugra} = \frac{\mathrm{SO}(5,21)}{\mathrm{SO}(5)\times\mathrm{SO}(21)}/\mathrm{SO}(5,21,\mathbb{Z}) \tag{7.9}$$

This moduli space corresponds to the geometric moduli of $K3$, the NS and RR potentials and the dilaton.

Six dimensional supergravity admits BPS black string solutions preserving 8 supercharges, charged under the 3-form field strengths. These solutions correspond to bound states of D5/NS5 branes wrapping the entire $K3$, D3 branes wrapping 2-cycles of $K3$ and F1/D1 strings. The charges of the black strings take values in the lattice $\Gamma^{5,21}$. The discrete U-duality group $\mathrm{SO}(5,21,\mathbb{Z})$ of the theory is equal to the automorphism group of the charge lattice. For any primitive lattice vector, there is always a U-duality transformation that can rotate it into a bound state of only D1 and D5 branes. The charge lattice $\Gamma^{5,21}$ can be embedded inside the vector space $W=\mathbb{R}^{5,21}$. Each point z on the moduli space (7.9) corresponds to a decomposition into positive and negative subspaces $W=V_+\oplus V_-$, so the moduli space of supergravity can be understood as the space of positive 5-planes inside $\mathbb{R}^{5,21}$.

[31] We would like to emphasize that we are not proposing that there is a well defined UV point for the CFT perturbed by irrelevant operators. If such a theory existed, it would be dual to asymptotically flat string theory. Instead we are treating the theory living on the branes as an effective field theory near the IR fixed point and consider Wilsonian RG flow towards the IR. We find that it is very constrained since there are only a finite number of irrelevant operators allowed by the symmetries. We claim that this self-consistent flow to the fixed point should be related to the attractor flow.

[32] It would be very interesting to study 4-dimensional black holes in $\mathcal{N}=2$ supergravity, where the attractor mechanism has a richer structure. In this case the near horizon geometry of an extremal black hole is $AdS_2\times S^2$. Unfortunately, the field theory side is not well understood. In general we would expect a $0+1$ dimensional theory which would flow in the 'IR' to some kind of superconformal quantum mechanics, leading to an $\mathrm{AdS}_2/\mathrm{CFT}_1$ duality. Since the precise meaning of the latter is still mysterious, even at the fixed point, it seems difficult to study the flow towards the fixed point with present technology.

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For a given charge vector Γ and given position on the moduli space we decompose $\Gamma = \Gamma_+ + \Gamma_-$ where $\Gamma_\pm \in V_\pm$. It can be shown that the central charge, or tension, of the black string is

$$Z(\Gamma, z) = |\Gamma_+| \tag{7.10}$$

Taking into account that

$$|\Gamma|^2 = |\Gamma_+|^2 - |\Gamma_-|^2 \tag{7.11}$$

is independent of the moduli z, we see that $|Z|$ is minimized when $\Gamma_- = 0$. This is equivalent to the set of positive 5-planes containing the vector Γ. It is not difficult to see that this attractor submanifold has locally the structure of the coset

$$\mathcal{M}^{*,\Gamma}_{\text{sugra}} = \frac{\text{SO}(4,21)}{\text{SO}(4) \times \text{SO}(21)} \tag{7.12}$$

The precise way in which this submanifold is embedded in the bigger space (7.9) depends on the charge vector Γ and can be easily determined using for example the analysis of [16].

The theory living on the branes is a 2-dimensional effective field theory, which flows in the IR to a 2d CFT with $\mathcal{N} = (4,4)$ supersymmetry. The supergravity attractor flow towards the AdS_3 throat should be dual to an RG flow of a 2d effective field theory towards a 2d CFT in the IR, at least near the fixed point. In other words, the theory on the brane, seen as an effective low energy theory, is a 2d CFT perturbed by irrelevant operators. The RG flow of this theory should be dual to the attractor flow in supergravity.

As we explained in the previous subsection, if we want to check this correspondence to first order we have to check three conditions. The fact that the moduli spaces in the IR are the same is a well known result [16], where we recognize that the space (7.12) is of the general form of the moduli space of $\mathcal{N} = (4,4)$ superconformal field theories (5.6). So the condition (7.7) is satisfied.

Let us now consider the second condition, which is the matching of the fixed moduli to the irrelevant operators which preserve the supersymmetry and R-symmetry. We want to perturb the CFT by irrelevant operators which do not break the $\mathcal{N} = (4,4)$ supersymmetry, but only the conformal invariance. Also we do not want to break the $\text{SO}(4)^R$ symmetry, which corresponds to the spherical symmetry around the black string. This question was discussed in [45, 46]. With these restrictions, as we explained in section 5.3 using the representation theory of the $\mathcal{N} = (4,4)$ algebra, the only candidate irrelevant operators are the descendants of chiral primaries ϕ_I of charge $(1,1)$. By acting with two supercurrents on each side we get $\text{SO}(4)^R$ neutral operators of conformal dimension $(2,2)$ of the form

$$\Phi_I = G^{-+}_{-1/2} G^{--}_{-1/2} \overline{G}^{-+}_{-1/2} \overline{G}^{--}_{-1/2} \cdot \phi_I \tag{7.13}$$

These are the only irrelevant operators preserving the $\mathcal{N} = (4,4)$ structure and which are $\text{SO}(4)^R$ singlets. In the notation of section 6.9 they can be written as

$$20\times \quad \alpha^{(1,1)}_{-2} |N-2\rangle \quad \text{and} \quad 1\times \quad \alpha^{(0,0)}_{-3} |N-3\rangle \quad \text{and} \quad 1\times \quad \alpha^{(2,2)}_{-1} |N-1\rangle \,. \tag{7.14}$$

The fact that the single-particle operators of this form are in one-to-one correspondence with the fixed moduli was already noted in [38]. There are 21+1 of them corresponding to

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the 21 fixed moduli of supergravity and the size of the 3-sphere. It is easy to check that the relation between masses and conformal dimension is correct.

These irrelevant operators are sections of a vector bundle as described in section 3. At the same time they describe motion away from the moduli space of conformal field theories $\mathcal{M}_{\rm CFT}$ into the bigger space $\mathcal{M}_{\rm QFT}$ of $\mathcal{N} = (4,4)$ quantum field theories. In this sense the bundle of the operators (7.14) is isomorphic the normal bundle $\mathcal{N}_{\rm CFT}$ of $\mathcal{M}_{\rm CFT}$ inside $\mathcal{M}_{\rm QFT}$. The connection on this bundle can be determined by the results of the previous sections about the connection for the chiral primaries ϕ and the supercurrents. It is not difficult to see that we have the following result

$$\mathcal{N}_{\rm CFT} = \mathcal{V}_{\mathbf{21}} \oplus \mathcal{V}_{\mathbf{1}} \tag{7.15}$$

Now from the supergravity side we have to compute the normal bundle of (7.12) inside (7.9). It is easy to see that it is exactly the same bundle $\mathcal{V}_{\mathbf{21}}$. If we add to it one more direction corresponding to increasing the size of the 3-sphere we have

$$\mathcal{N}_{\rm sugra} = \mathcal{V}_{\mathbf{21}} \oplus \mathcal{V}_{\mathbf{1}} \tag{7.16}$$

So we find precise agreement between the two normal bundles, showing that the last condition (7.8) is also satisfied.[33] This shows that to first order away from the fixed point the attractor flow agrees with RG-flow on the boundary.

7.4 Finite flows

A natural question is whether we can extend the previous arguments to higher orders in perturbation theory towards the UV. As we explained before it is hard to give a precise UV completion of the CFT perturbed by irrelevant operators, which is related to the absence of decoupling between open and closed strings away from the $\alpha' \to 0$ limit. Despite these problems let us describe briefly what the full attractor flow for the D1/D5 system looks like on the supergravity side. These solutions where discussed in detail in [47].

The metric has the form

$$ds^2 = e^{2U(r)}(-dt^2 + dx^2) + e^{-2U(r)}(dr^2 + r^2 d\Omega_3^2) \tag{7.17}$$

We take the moduli at infinity to be at a general point $z^\infty \in \mathcal{M}_{\rm sugra}$, which corresponds to a specific orientation of the positive 5-plane V_+^∞ inside the space $\mathbb{R}^{5,21}$. We also choose a charge vector Γ, which does not generally lie inside V_+^∞. As we move towards the black string the orientation of the 5-plane will change and at the attractor point it will be such that $\Gamma \in V_+^*$. To fully specify the solution we need to determine the function $U(r)$ and the orientation of the 5-plane as a function of the radius $V_+(r)$.

It turns out that the solutions are very simple. We decompose the charge vector Γ into its projections on the positive and negative subspaces at infinity which gives two vectors $\Gamma_\pm^\infty \in \mathbb{R}^{5,21}$. These two vectors define a fixed 2-plane K inside $\mathbb{R}^{5,21}$. Now, the radial

[33] We should emphasize that the agreement between equations (7.15),(7.16) does not only refer to the rank of the bundles but to the full geometry of the bundle over the moduli space.

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dependence of the orientation of the 5-plane $V_+(r)$ is given the action of an SO(5, 21) boost $\mathcal{B}_K(\psi)$ along the constant 2-plane K, which is rotating the Γ^- component into the Γ^+ and with r-dependent rapidity $\psi(r)$. So we have

$$V_+(r) = \mathcal{B}_K(\psi) \cdot V_+^\infty \tag{7.18}$$

where at infinity we must have $\psi(\infty) = 0$ to satisfy the boundary conditions, while near the horizon ψ must take a value such that $\Gamma \in V_+$ to satisfy the attractor condition $\Gamma^- = 0$. All the information about the solution is contained in the two functions $U(r)$ and $\psi(r)$. In units where the 6d Planck length is one, the two functions are

$$\begin{aligned} e^{-2U(r)} &= \left(1 + \frac{|\Gamma_+^\infty| - |\Gamma_-^\infty|}{r^2}\right)^{1/2} \left(1 + \frac{|\Gamma_+^\infty| + |\Gamma_-^\infty|}{r^2}\right)^{1/2} \\ e^{\psi(r)} &= \sqrt{\frac{|\Gamma_+^\infty| - |\Gamma_-^\infty| + r^2}{|\Gamma_+^\infty| + |\Gamma_-^\infty| + r^2}} \end{aligned} \tag{7.19}$$

From these one can reconstruct the full solution, including the radial dependence of $\Gamma_\pm(r)$ and of the 3-form field strengths following the detailed analysis in [47]. As an easy check we can see that the ADM mass of this solution is indeed proportional to $|\Gamma_+^\infty|$ as expected from (7.10), while in the near horizon region we get an AdS_3 throat of size proportional to $|\Gamma| = \sqrt{|\Gamma_+^\infty|^2 - |\Gamma_-^\infty|^2}$, which is independent of the value of the moduli at infinity.

Notice that the motion on the moduli space $\mathcal{M}_{\text{sugra}}$ from z^∞ to $z^* \in \mathcal{M}_{\text{sugra}}^{*,\Gamma}$ is rather simple and given by the action of a one-parameter group of SO(5, 21) Lorentz boosts along a constant 2-plane (7.18). We take the simplicity of the solution as an indication that the corresponding RG-flow, appropriately interpreted, at finite scales might be also simple.

One approach would be to try to apply tt^* inspired arguments away from the conformal point. As we saw, the set of irrelevant operators preserving the $\mathcal{N} = (4, 4)$ supersymmetry is finite, so it is not totally inconceivable that by generalizing the tt^* formalism we might be able to find RG-flow orbits in this restricted subset of parameters. Ideally we would like to reproduce the full moduli space (7.9) from the perturbed $\mathcal{N} = (4, 4)$ and the attractor flows described above. The tt^* formalism has already been used in theories away from criticality. The reason that we cannot apply the standard tt^* arguments directly to our system is that the irrelevant operators that we are perturbing by are of the form (7.13). The tt^* formalism is based on the $\mathcal{N} = (2, 2)$ algebra. From an $\mathcal{N} = (2, 2)$ point of view, the operators (7.13) are not F-term perturbations, since they involve too many supercurrents, which are not visible in a single $\mathcal{N} = (2, 2)$ subalgebra, and naively should not be protected. It is the underlying $\mathcal{N} = (4, 4)$ which protects these operators. It would be very interesting to generalize the tt^* framework for perturbations of this form in $\mathcal{N} = (4, 4)$ theories.

Another way to study finite flows away from the conformal fixed point would be to go to higher orders in conformal perturbation theory. Since we have included all irrelevant operators that preserve $\mathcal{N} = (4, 4)$ supersymmetry, in a scheme in which these supersymmetries are preserved no further irrelevant operators should be generated in the effective action, as these would necessarily break some of the supersymmetries. Therefore, in such a scheme all the conformal perturbation theory would do is to generate a non-trivial scale

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dependence of the irrelevant couplings. One can imagine that the latter may eventually be related to the rather simple form of the flow solution (7.19) and it would be interesting to explore this further.

7.5 A decoupling limit and 6d gauge theory

Finally, let us mention that certain orbits of the attractor flow can be embedded in a boundary theory with an honest decoupling limit in the following way, which was also described in [48]. Consider IIB compactified on $K3$ of volume $V_{K3} = v\alpha'^2$ with v dimensionless, and a bound state of D1/D5 branes. The D1/D5 solution is

$$\begin{aligned} ds^2 &= Z_1^{-1/2} Z_5^{-1/2}(-dt^2 + dx^2) + Z_1^{1/2} Z_5^{1/2}(dr^2 + r^2 d\Omega_3^2) + Z_1^{1/2} Z_5^{-1/2} \sqrt{v}\alpha' ds_{K3}^2 \\ e^{2\phi} &= g_s^2 Z_1 Z_5^{-1} \\ Z_1 &= 1 + \frac{g_s Q_1 \alpha'/v}{r^2} \\ Z_5 &= 1 + \frac{g_s Q_5 \alpha'}{r^2} \end{aligned} \tag{7.20}$$

where ds_{K3}^2 is the metric of a $K3$ of unit volume. The standard decoupling limit which leads to $\mathrm{AdS}_3/\mathrm{CFT}_2$ is $\alpha' \to 0$ keeping g_s, v constant. Instead we consider the decoupling limit corresponding to a D5 brane in flat space

$$\alpha' \to 0, \qquad g_s \alpha' = g_{\mathrm{YM}}^2 = \text{const}, \qquad V_{K3} = v\alpha'^2 = \text{const}, \qquad U = \frac{r}{\alpha'} = \text{const}. \tag{7.21}$$

In this limit we do have a decoupling of the open and closed modes. Also, the $+1$ drops out of the harmonic function Z_5 but not Z_1

$$Z_1 = 1 + \frac{g_{\mathrm{YM}}^2/V_{K3}}{U^2}, \qquad Z_5 \simeq \frac{g_{\mathrm{YM}}^2 Q_5}{\alpha'^2 U^2}. \tag{7.22}$$

The decoupled supergravity solution takes the form

$$\begin{aligned} \frac{ds^2}{\alpha'} &= \frac{U}{\sqrt{g_{\mathrm{YM}}^2 Q_5}} Z_1^{-1/2}(-dt^2 + dx^2) \\ &\qquad + \frac{\sqrt{g_{\mathrm{YM}}^2 Q_5} Z_1^{1/2}}{U}(dU^2 + U^2 d\Omega_3^2) + \frac{U}{\sqrt{g_{\mathrm{YM}}^2 Q_5}} Z_1^{1/2} \sqrt{V_{K3}} ds_{K3}^2, \\ e^{2\phi} &= \frac{g_{\mathrm{YM}}^2 U^2}{Q_5} Z_1. \end{aligned} \tag{7.23}$$

It is easy to see that this is asymptotically locally the same as the decoupling limit of the D5 brane in flat space, but the global structure is $\mathbb{R}^{1,1} \times K3$. The dilaton blows up at infinity but we can S-dualize to the NS5 brane solution which is well behaved there. As we move towards the IR the size of the $K3$ shrinks and reaches a stringy size fixed by the attractor mechanism, while the rest of the geometry becomes $\mathrm{AdS}_3 \times \mathrm{S}^3$. From the point of view of our general solution (7.19) this corresponds to scaling $\Gamma_\pm^\infty \to \infty$ in such a way

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that the +1 in the second harmonic function in the expression for $e^{-2U(r)}$ can be dropped but not in the first.

Holographically in the UV we start with the 5+1 dimensional NS5 brane $(1,1)$ little string theory living on $\mathbb{R}^{1,1} \times K3$. Below energy scales of the order $(g_{\rm YM})^{-1}$ the theory can be well described by 5+1 SYM on $\mathbb{R}^{1,1} \times K3$. At energies below $(V_{K3})^{-1/4}$ we can integrate out the $K3$ modes and end up with the 2-dimensional D1/D5 SCFT in the IR. Along this RG-flow between a 5+1 and a 1+1 theory the scalar moduli flow and get fixed values by the attractor mechanism. So in principle the RG flow between 5+1 dimensional SYM on $\mathbb{R}^{1,1} \times K3$ and the 2d CFT in the IR should contain a holographic description of the attractor mechanism for this simple system, at least for some attractor flows. This is hard to study in general but it would be intersecting to see if it is possible to truncate the RG-flow to the BPS sector of the system, by identifying the operators in the gauge theory which flow to the chiral primaries in the IR and studying the supersymmetric sector of the RG-flow. We identified the corresponding operators in section 6.10 but leave the study of the boundary RG-flow for future work.

8 Black hole Berry phase

Finally we would like to mention one more application of our analysis. We have computed the connection for the chiral primary operators in the NS sector of the $\mathcal{N} = (4,4)$ D1/D5 SCFT. By spectral flow the chiral primaries are related to Ramond ground states. This means that we know the exact connection for the vector bundle of Ramond ground states over the moduli space of the theory. In spacetime the Ramond ground states correspond to quantum microstates of a bound state of D1 and D5 branes, wrapped around $S^1 \times K3$, which is a small black hole in 5d. The connection on the bundle of chiral primaries is telling us how different microstates of the black hole mix as we move on the moduli space. This is is a version of the (nonabelian) Berry phase [49, 50] for the internal states of the black hole, under adiabatic change of the moduli of the compactification. In principle, this exactly computable holonomy would allow one to set up interference experiments sensitive to the internal microstate of the black hole. Obviously preparing a black hole in a pure state in practice would be highly challenging. It would be interesting to explore the implications of this phenomenon in more detail, we hope to report on it in the future.

Other systems in string theory where Berry's phase appears and has interesting interpretation have been studied recently [51–53].

9 Summary and further directions

The main technical point of this paper was the analysis of the moduli dependence of the chiral ring for $\mathcal{N} = (4,4)$ superconformal field theories. It was based on an application of the tt^* equations which we derived[34] from general principles of conformal perturbation theory and not relying on the topological twisting. This derivation clarifies the connection between the work based on topological-antitopological fusion [3, 4] and that on standard

[34] Of course the original derivation is more general, it also works for non-conformal $\mathcal{N} = (2,2)$ theories.

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CFT arguments [25, 26]. The main result is that for $\mathcal{N} = (4,4)$ theories the chiral ring is covariantly constant over the moduli space. We found that the bundles of chiral primaries are constrained to be homogeneous bundles, whose curvature is exactly computable.

In the case of $\mathrm{AdS}_3/\mathrm{CFT}_2$ our results imply a non-renormalization theorem for 3-point functions of chiral primaries and more general extremal correlators, even at finite values of N. This explains the agreement found in [5–8]. To gain a better understanding of the relation between different points on the moduli space it would be useful to clarify the global structure of the moduli space of the SCFT and possible monodromies of the chiral ring around singularities.

The connection for the chiral primaries that we computed in this paper can be used to demonstrate agreement between the attractor flow and RG-flow in the vicinity of the fixed point, in the simple case of an infinite D1/D5 black string. It would be interesting to extend this analysis to finite order away from the fixed point, for example under the flow by the irrelevant operators mentioned in the text which do not break the $\mathcal{N} = (4,4)$ supersymmetry. This is a finite set of operators so it might be possible to find constrained self-consistent flows towards the UV related to the attractor flows in supergravity. In particular, since we are in a certain sense studying the BPS sector of the theory, we might hope to reconstruct the full geometry of the supergravity moduli space (7.9) from the geometry of the field theory moduli space away from criticality.

An obvious generalization would be to set up a similar analysis for systems with less supersymmetry. One example is the $\mathcal{N} = (0,4)$ MSW superconformal field theory which appears on the worldvolume of an M5 brane wrapping a four-cycle in a Calabi-Yau manifold [17]. The five-dimensional supergravity solution has an $\mathrm{AdS}_3 \times \mathrm{S}^2$ near horizon geometry and has a more interesting attractor flow towards the fixed point. One could try to identify the constraints from supersymmetry on the structure of the moduli spaces and the chiral ring. Moreover, this theory has a very interesting set of supergravity solutions [54] corresponding to multi-centered black holes which can be constructed by perturbing the theory towards the IR. It would be nice to see if the structure found from supergravity can be reproduced in any sense from the RG-flow in the boundary theory.

Four-dimensional black holes in $\mathcal{N} = 2$ supergravity provide another interesting example where a suitable extension of our results might be obtainable. In this case the theory on the branes should flow to "superconformal quantum mechanics" which would be the boundary side of $\mathrm{AdS}_2/\mathrm{CFT}_1$. This conjectured duality has not been fully understood so it is not straightforward to make progress in this direction.

It would also be interesting to understand how to formulate the computation of Berry's phase for the microstates of other supersymmetric black holes. Again the $\mathcal{N} = 2$ 4d case would be most interesting, but difficult for the reasons mentioned in the previous paragraph. It might be interesting to see if anything can be said about states in the D1-D5 system which are of the form chiral primary-anything, corresponding to D1-D5-P microstates. It is not clear if the holonomy for such states is sufficiently constrained by supersymmetry, but as these would correspond to microstates of a 5d black hole with a macroscopic horizon they are worthwhile to investigate.

Finally let us mention another direction which might be interesting to explore further.

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While the connection for chiral primaries over the moduli space has been studied in detail for the case of 2d superconformal field theories, the same analysis has not been performed for their higher dimensional analogues. More precisely, one could try to study the connection for the operators in the chiral ring of 4d superconformal gauge theories. In particular it would be interesting to see if there is any way of deriving equations similar to tt^* for 4 dimensional theories, expressing the curvature of the bundle of chiral primaries in terms of the chiral ring coefficients. If such relations exist, they may lead to interesting constraints for the Kähler metric on the moduli space of $\mathcal{N} = 1$ SCFTs and they may be useful for the analysis of aspects of Seiberg duality in $\mathcal{N} = 1$ theories.

Let us close with a simple observation in this direction. Consider four dimensional $\mathcal{N} = 4$ SU(N) SYM at the superconformal point, whose R-symmetry is SO(6). This theory is not an isolated conformal field theory since we can continuously vary the coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\rm YM}^2}$ without breaking conformal invariance. Its moduli space $\mathcal{M}$ is the upper half-plane modded out by the action of a certain subgroup of the SL$(2,\mathbb{Z})$ duality group. Operators in short representations can be constructed starting with a holomorphic combination of two of the six real scalars of the theory, say $Z = \Phi^1 + i\Phi^2$, and then considering operators of the form $\mathrm{Tr}Z^p$ and their products. By acting on these operators with the supercharges and momentum generators we can construct the full superconformal multiplet. Motion along $\mathcal{M}$ is generated by marginal operators which in four dimensions have conformal dimension 4. In $\mathcal{N} = 4$ these marginal operators can be written as descendants of chiral primaries in the form

$$\mathcal{O} = Q^4 \mathrm{Tr} Z^2 \tag{9.1}$$

This is a complex operator whose real and imaginary parts express the coupling of the Lagrangian density to $\frac{1}{g_{\rm YM}^2}$ and θ respectively. In components

$$\mathcal{O} \sim \mathrm{Tr}(F_{\mu\nu}^2) + i\mathrm{Tr}(F \wedge F) + \ldots \tag{9.2}$$

The metric on the moduli space $\mathcal{M}$ is given by the following expression

$$ds^2 = g_{\tau\overline{\tau}} d\tau d\overline{\tau} \sim \frac{1}{(\mathrm{Im}\tau)^2} d\tau d\overline{\tau} \tag{9.3}$$

and is related to the 2-point function

$$\langle \mathcal{O}(x) \overline{\mathcal{O}}(y) \rangle = \frac{g_{\tau\overline{\tau}}}{|x-y|^8} \tag{9.4}$$

The important point is that this metric is not flat. Hence the tangent bundle $\mathcal{TM}$ has nonzero curvature. The marginal operators (9.1) correspond to tangent vectors on $\mathcal{M}$. Then under parallel transport on the moduli space the marginal operators will mix as

$$\mathcal{O} \to e^{i\chi}\mathcal{O}, \qquad \overline{\mathcal{O}} \to e^{-i\chi}\overline{\mathcal{O}} \tag{9.5}$$

where the angle χ is exactly computable from the geometry of the moduli space (9.3). From (9.1) we see that the marginal operators are sections of a bundle which is the tensor product of the bundle of the supercharges and the bundle whose fiber is generated by

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the chiral primary $\mathrm{Tr}Z^2$. As in the two-dimensional case, we expect that the SO(6) R-symmetry is covariantly constant over the moduli space $\mathcal{M}$. Then the chiral primary $\mathrm{Tr}Z^2$ cannot get a phase under parallel transport. Thus we learn that the phase (9.5) is coming from a mixing of the supercharges which corresponds to a rotation under the U(1) outer automorphism of the $\mathcal{N} = 4$ algebra in 4d.[35] This mixing is exactly computable at all values of the coupling from the geometry of the moduli space. In this case it seems that we only have curvature for the supercharges and not the chiral primaries.[36]

It would be interesting to explore the constraints from supersymmetry on the geometry of the chiral ring over the moduli space for other four dimensional superconformal field theories, with less supersymmetry or for other operators in short multiplets such as the 1/16 BPS operators in $\mathcal{N} = 4$.

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A The superconformal algebra

A.1 The $\mathcal{N} = 2$ superconformal algebra

The $\mathcal{N} = 2$ algebra has the form

$$\begin{aligned}
T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots \\
J(z)J(w) &= \frac{c/3}{(z-w)^2} + \dots \\
T(z)J(w) &= \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w} + \dots \\
G^+(z)G^-(w) &= \frac{2c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w) + \partial J(w)}{z-w} + \dots \\
T(z)G^\pm(w) &= \frac{3}{2}\frac{G^\pm(w)}{(z-w)^2} + \frac{\partial G^\pm(w)}{z-w} + \dots \\
J(z)G^\pm(w) &= \pm\frac{G^\pm(w)}{z-w} + \dots
\end{aligned} \tag{A.1}$$

[35] This is the $\mathrm{U}(1)_Y$ "bonus symmetry" discussed in [9].

[36] We do however have curvature for the descendants of the chiral primaries due to the curvature of the supercharges.

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and

$$T^\dagger = T, \qquad J^\dagger = J, \qquad (G^\pm)^\dagger = G^\mp \tag{A.2}$$

We define the modes

$$\begin{aligned} L_n &= \frac{1}{2\pi i}\oint z^{n+1}T(z)dz \\ G_r^\pm &= \frac{1}{2\pi i}\oint z^{r+1/2}G^\pm(z)dz \\ J_n &= \frac{1}{2\pi i}\oint z^{n}J(z)dz \end{aligned} \tag{A.3}$$

and we have the commutation relations

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\,\delta_{m+n,0} \\ [J_m, J_n] &= \frac{c}{3}m\,\delta_{m+n,0} \\ [L_m, J_n] &= -nJ_{m+n} \\ \{G_r^-, G_s^+\} &= 2L_{r+s} - (r-s)J_{r+s} + \frac{c}{3}(r^2-1/4)\,\delta_{r+s,0} \\ \{G_r^+, G_s^+\} &= \{G_r^-, G_s^-\} = 0 \\ [L_m, G_r^\pm] &= (m/2-r)G_{m+r}^\pm \\ [J_m, G_r^\pm] &= \pm G_{m+r}^\pm \end{aligned} \tag{A.4}$$

where r, s is half-integer in the NS sector and integer in the R sector, and have the following hermiticity conditions

$$(L_m)^\dagger = L_{-m}, \qquad (J_m)^\dagger = J_{-m}, \qquad (G_r^\pm)^\dagger = G_{-r}^\mp \tag{A.5}$$

A.2 The $\mathcal{N} = 4$ superconformal algebra

In the small $\mathcal{N} = 4$ algebra the bosonic currents are $T(z), J^i(z), i = 1, 2, 3$ and the supercurrents $G^{\pm+}(z)$ and $G^{\pm-}(z)$. The central charge and the level are related by $c = 6k$. The algebra has the following form

$$\begin{aligned} T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots \\ J^i(z)J^j(w) &= \frac{k}{2}\frac{\delta^{ij}}{(z-w)^2} + i\epsilon^{ijk}\frac{J^k(w)}{z-w} + \dots \\ T(z)J^i(w) &= \frac{J^i(w)}{(z-w)^2} + \frac{\partial J^i(w)}{z-w} + \dots \\ T(z)G^{ab}(w) &= \frac{3}{2}\frac{G^{ab}(w)}{(z-w)^2} + \frac{\partial G^{ab}(w)}{z-w} + \dots \\ J^i(z)G^{a\pm}(w) &= \frac{1}{2}\sigma^i_{ba}\frac{G^{b\pm}(w)}{(z-w)} + \dots \end{aligned} \tag{A.6}$$

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and

$$
\begin{aligned}
G^{a+}(z)(G^{b+})^\dagger(w) &= \frac{2c}{3}\frac{\delta_{ab}}{(z-w)^3} + \frac{4\overline{\sigma}^i_{ab}J^i}{(z-w)^2} + \frac{2T(w)\delta_{ab}}{(z-w)} + \frac{2\overline{\sigma}^i_{ab}\partial J^i}{z-w} + \dots \\
G^{a+}(z)G^{b+}(w) &= \text{regular} \\
G^{a-}(z)G^{b-}(w) &= \text{regular}
\end{aligned}
\tag{A.7}
$$

where $a, b = +, -$ and σ^i_{ab} are the Pauli matrices. The hermiticity conditions of the generators are

$$
T^\dagger = T, \qquad (J^i)^\dagger = J^i, \qquad (G^{++})^\dagger = G^{--}, \qquad (G^{+-})^\dagger = -G^{-+} \tag{A.8}
$$

B Some useful OPEs for $\mathcal{N} = (2,2)$

Let us call ϕ a (cc) field of $(L_0, J_0) = (h, q)$. We have the following OPEs

$$
\begin{aligned}
G^+(z)\phi(w) &= \text{regular} \\
G^-(z)\phi(w) &= \frac{(G^-_{-1/2}\cdot\phi)(w)}{z-w} + \dots \\
T(z)\phi(w) &= h\frac{\phi(w)}{(z-w)^2} + \frac{\partial\phi(w)}{z-w} + \dots \\
J(z)\phi(w) &= q\frac{\phi(w)}{z-w} + \dots
\end{aligned}
\tag{B.1}
$$

Using the algebra and that $h = q/2$ for a chiral primary we find

$$
G^+(z)(G^-_{-1/2}\cdot\phi)(w) = 2q\frac{\phi(w)}{(z-w)^2} + 2\frac{\partial\phi(w)}{z-w} + \dots \tag{B.2}
$$

For chiral primaries with $(h, q) = (1/2, 1)$ this becomes

$$
G^+(z)(G^-_{-1/2}\cdot\phi)(w) = 2\partial_w\left(\frac{\phi(w)}{z-w}\right) + \dots \tag{B.3}
$$

C Curvature of supercurrents in $\mathcal{N} = (2,2)$

We have to study the 4-point function of the form (4.10). For definiteness we will consider

$$
A = \langle \mathcal{O}_i(x)\overline{\mathcal{O}_j}(y)G^+(z)G^-(w)\rangle \tag{C.1}
$$

As a function of z, A is holomorphic so it is determined by its singularity structure at $z = x, y, w$. For this we need the OPEs of $G^+(z)$ with the other insertions. We have the following results

$$
\begin{aligned}
G^+(z)G^-(w) &= \frac{2c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w)+\partial J(w)}{z-w} + \dots \\
G^+(z)\mathcal{O}_i(x) &= \partial_x\left(\frac{(\overline{G}^-_{-1/2}\cdot\phi_i)(x)}{z-x}\right) + \dots \\
G^+(z)\overline{\mathcal{O}_j}(y) &= \text{regular}
\end{aligned}
\tag{C.2}
$$

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where we used $\mathcal{O}_i = \frac{1}{2}G^-_{-1/2}\overline{G}^-_{-1/2}\cdot\phi_i$, $\overline{\mathcal{O}_j} = \frac{1}{2}G^+_{-1/2}\overline{G}^+_{-1/2}\cdot\overline{\phi_j}$ and the $\mathcal{N}=(2,2)$ algebra. So we have

$$\begin{aligned} A =& \langle \mathcal{O}_i(x)\overline{\mathcal{O}_j}(y)\left(\frac{2c/3}{(z-w)^3}+\frac{2J(w)}{(z-w)^2}+\frac{2T(w)+\partial J(w)}{z-w}\right)\rangle \\ &+\partial_x\left(\frac{1}{z-x}\langle(\overline{G}^-_{-1/2}\cdot\phi_i)(x)\overline{\mathcal{O}_j}(y)G^-(w)\rangle\right) \end{aligned} \tag{C.3}$$

This is of the form $A = A_1 + A_2$ where each term corresponds to one of the lines in the expression above. The term A_1 can be easily evaluated by the usual conformal Ward identities on the correlation function

$$\langle \mathcal{O}_i(x)\overline{\mathcal{O}_j}(y)\rangle = \frac{g_{i\overline{j}}}{|x-y|^4} \tag{C.4}$$

After some algebra we find

$$A_1 = \frac{2cg_{i\overline{j}}}{3|x-y|^4(z-w)^3} + \frac{2g_{i\overline{j}}}{(w-x)^2(w-y)^2(\overline{x}-\overline{y})^2(z-w)} \tag{C.5}$$

To compute A_2 we need the correlation function

$$B = \langle(\overline{G}^-_{-1/2}\cdot\phi_i)(x)\overline{\mathcal{O}_j}(y)G^-(w)\rangle \tag{C.6}$$

As a function of w the expression B is holomorphic, so again we can use the OPEs to determine it. We have

$$B = \partial_y\left(\frac{1}{w-y}\langle(\overline{G}^-_{-1/2}\cdot\phi_i)(x)(\overline{G}^+_{-1/2}\cdot\overline{\phi_j}(y)\rangle\right) - \frac{2}{w-x}\langle\mathcal{O}_i(x)\overline{\mathcal{O}_j}(y)\rangle \tag{C.7}$$

Now using

$$\langle(\overline{G}^-_{-1/2}\cdot\phi_i)(x)(\overline{G}^+_{-1/2}\cdot\overline{\phi_j}(y)\rangle = 2\frac{g_{i\overline{j}}}{(x-y)(\overline{x}-\overline{y})^2} \tag{C.8}$$

and expression (C.4) we find

$$B = -\frac{2g_{i\overline{j}}}{(w-x)(w-y)^2(\overline{x}-\overline{y})^2} \tag{C.9}$$

so

$$A_2 = \frac{2g_{i\overline{j}}(w-2x+z)}{(w-x)^2(w-y)^2(\overline{x}-\overline{y})^2(x-z)^2} \tag{C.10}$$

Finally going back to (C.3) we can compute $A = A_1 + A_2$ and we find

$$A = \frac{2cg_{i\overline{j}}}{3|x-y|^4(z-w)^3} + \frac{2g_{i\overline{j}}}{(w-y)^2(z-w)(x-z)^2(\overline{x}-\overline{y})^2} \tag{C.11}$$

Similarly we can compute the other 4-point functions needed for the computation of the curvature of the supercurrents.

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D 4-point functions in $\mathcal{N} = (2,2)$

Consider a (cc) field ϕ_k and an (aa) $\overline{\phi_l}$. We want to simplify the 4-point function

$$G(x,y,z,w) = \langle \mathcal{O}_i(x)\overline{\mathcal{O}_j}(y)\phi_k(z)\overline{\phi_l}(w)\rangle \tag{D.1}$$

where the marginal operators are descendants of the chiral ring

$$\mathcal{O}_i = \frac{1}{2}G^-_{-1/2}\overline{G^-}_{-1/2}\cdot\phi_i, \qquad \overline{\mathcal{O}_j} = \frac{1}{2}G^+_{-1/2}\overline{G^+}_{-1/2}\cdot\overline{\phi_j} \tag{D.2}$$

We can also write the operators as [32]

$$\mathcal{O}_i(x) = \frac{1}{2}\frac{1}{2\pi i}\oint_x ds\frac{s-t}{x-t}G^-(s)\left(\overline{G}^-_{-1/2}\phi_i\right)(x) \tag{D.3}$$

we choose $t = z$ and we deform the contours. The supercurrent $G^-(s)$ annihilates $\overline{\phi_{\overline{l}}}$ and it has a first order pole with $\phi_k(z)$ which is cancelled with the $(s-z)$ in the numerator. Finally we have to use the $\mathcal{N} = (2,2)$ algebra to compute its OPE with the insertion at y. We find that the answer is

$$G(x,y,z,w) = \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{y-z}{x-z}\langle\left(\overline{G}^-_{-1/2}\phi_i\right)(x)\left(\overline{G}^+_{-1/2}\overline{\phi_j}\right)(y)\phi_k(z)\overline{\phi_l}(w)\rangle\right) \tag{D.4}$$

Doing the same for the supercurrent $\overline{G}^+$ we end up with the the following expression

$$G(x,y,z,w) = \partial_y\partial_{\overline{y}}\left(\frac{|y-z|^2}{|x-z|^2}\langle\phi_i(x)\overline{\phi_j}(y)\phi_k(z)\overline{\phi_l}(w)\rangle\right) \tag{D.5}$$

E OPE between chiral primary and antichiral primary

Consider (cc) field ϕ_i of charge $q_i > 0$ and (aa) field $\overline{\phi_l}$ of charge $q_l < 0$ with $q_i < |q_l|$. Consider their OPE

$$\phi_i(z)\overline{\phi_l}(w) = \sum_{r\overline{r}}\frac{D^\rho_{i\overline{l}}A_\rho(z)}{(z-w)^r(\overline{z}-\overline{w})^{\overline{r}}} \tag{E.1}$$

The field A_ρ has U(1) charge $q_\rho = q_i + q_l < 0$, and conformal dimension $h_\rho = h_i + h_l - r$ (similarly for the right-moving side). From unitarity we have the condition $h_\rho \geq |q_\rho|/2$. Equivalently this means

$$r \leq q_i \tag{E.2}$$

If the inequality is saturated (and similarly on the right moving side) the corresponding field A_ρ will be antichiral primary of charge $q_i + q_l < 0$. So the OPE will have the form

$$\phi_i(z)\overline{\phi_l}(w) = \frac{D^{\overline{k}}_{i\overline{l}}\overline{\phi_k}(w)}{(z-w)^{q_i}(\overline{z}-\overline{w})^{\overline{q}_i}} + \ldots \tag{E.3}$$

The coefficients $D^{\overline{k}}_{i\overline{l}}$ are related to the chiral ring structure constants. We consider the 3-point function

$$\langle\phi_i(z)\overline{\phi_l}(w)\phi_n(y)\rangle \tag{E.4}$$

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and take the OPE in two different ways to show that

$$D^{\overline{k}}_{i\overline{l}} = C^m_{in} g^{\overline{k}n} g_{m\overline{l}} \tag{E.5}$$

So the conclusion is that the leading term of the OPE of (cc) with (aa) is given by the conjugated chiral ring coefficients.

F Contours

Now we want to study the first term of (4.28) using the OPE between ϕ_i and $\overline{\phi_j}$. We define

$$C = -\frac{1}{(2\pi)^2} \lim_{|r|\to 1} \int_{|y|=1} d\theta_1 \int_{|x|=r} d\theta_2 \left(r^2 \langle \phi_i(x)\overline{\phi_j}(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle - \langle \overline{\phi_j}(x)\phi_i(y)\phi_k(\infty)\overline{\phi_l}(0)\rangle \right) \tag{F.1}$$

We change the angle variables to $\theta = \frac{(\theta_1+\theta_2)}{2}$ and $\psi = \frac{(\theta_1-\theta_2)}{2}$ and we have

$$\begin{aligned} C = -\frac{2}{(2\pi)^2} \lim_{|r|\to 1} \int d\theta \int d\psi \Big(& r^2 \langle \phi_k(\infty)\overline{\phi_j}(e^{i(\theta+\psi)})\phi_i(re^{i(\theta-\psi)})\overline{\phi_l}(0)\rangle \\ & - \langle \phi_k(\infty)\phi_i(e^{i(\theta+\psi)})\overline{\phi_j}(re^{i(\theta-\psi)})\overline{\phi_l}(0)\rangle \Big) \end{aligned} \tag{F.2}$$

For $\psi \neq 0$ the contribution from ψ cancels with that from $-\psi$ in the limit $r \to 1$. However this does not mean that the integral is zero, since we may have δ-function-like contributions from $\psi = 0$. These contributions can be evaluated using the OPE of ϕ_i with $\overline{\phi_j}$ which is

$$\phi_i(z)\overline{\phi_j}(w) = \sum_\rho \frac{D^\rho_{i\overline{j}} A_\rho(w)}{(z-w)^{1-h_\rho}(\overline{z}-\overline{w})^{1-\overline{h}_\rho}} \tag{F.3}$$

Let us assume that the operator A_ρ has dimension $(h_\rho, h_\rho - s_\rho)$ where s_ρ is the spin. Then

$$C = -\frac{2}{(2\pi)^2} \lim_{r\to 1} \int d\theta \int_{-\delta}^{+\delta} d\psi \sum_\rho \frac{D_{\overline{l}\rho,k} D^\rho_{i\overline{j}}}{(z_1-z_2)^{1-h_\rho}(\overline{z}_1-\overline{z}_2)^{1-\overline{h}_\rho}} \left(\frac{(-1)^{s_\rho}|z_2|^2}{z_1^{h_\rho}\overline{z}_1^{\overline{h}_\rho}} - \frac{1}{z_2^{h_\rho}\overline{z}_2^{\overline{h}_\rho}} \right) \tag{F.4}$$

where $z_1 = e^{i(\theta+\psi)}$, $z_2 = re^{i(\theta-\psi)}$ and where δ is a small number that is kept constant as $\epsilon \to 0$. We can rewrite this as

$$C = -\frac{2}{(2\pi)^2} \lim_{r\to 1} \int d\theta \int_{-\delta}^{+\delta} d\psi \sum_\rho \frac{D_{\overline{l}\rho,k} D^\rho_{i\overline{j}}}{|1-re^{-2i\psi}|^{2-2h_\rho}(1-re^{2i\psi})^{s_\rho}} \left((-1)^{s_\rho} r^2 - \frac{e^{2is_\rho\psi}}{r^{h_\rho+\overline{h}_\rho}} \right) \tag{F.5}$$

If A_ρ is a spin zero field ($h_\rho = \overline{h}_\rho$), then the contribution is proportional to

$$\lim_{r\to 1} \int_{-\delta}^{+\delta} d\psi \frac{1}{|1-re^{-2i\psi}|^{2-2h_\rho}} \left(r^2 - \frac{1}{r^{2h_\rho}} \right) \tag{F.6}$$

One can show that this quantity[37] is finite and δ independent if $h_\rho = 0$ and zero if $h_\rho > 0$. Its value for $h_\rho = 0$ is

$$\lim_{r\to 1} \int_{-\delta}^{+\delta} d\psi \frac{r^2-1}{|1-re^{-2i\psi}|^2} = -\pi \tag{F.7}$$

[37] In fact, the integral in (F.4) can be explicitly evaluated for fixed r and with $\delta = \pi/2$ in terms of hypergeometric functions, but we will not present these expressions here.

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So from the spin zero fields only the identity operator will contribute to C a factor of

$$g_{i\bar{j}}g_{k\bar{l}} \tag{F.8}$$

Similarly we can show that from fields with nonzero spin, only $(1,0)$ and $(0,1)$ fields contribute. For the first case we need

$$\phi_i(z)\overline{\phi_j}(w) = \ldots + \frac{D^J_{i\bar{j}}J(w)}{(\overline{z}-\overline{w})} + \ldots \tag{F.9}$$

The coefficient $D^J_{i\bar{j}}$ can be easily computed using the Ward identities for J and we find

$$D^J_{i\bar{j}} = \frac{3}{c}g_{i\bar{j}} \tag{F.10}$$

where we used

$$\langle J(0)J(\infty)\rangle = \frac{c}{3}, \qquad \langle \phi_i(1)\overline{\phi_j}(0)J(\infty)\rangle = g_{i\bar{j}} \tag{F.11}$$

for fields $\phi_i, \overline{\phi_j}$ of charge $+1, -1$. Similarly

$$\langle \overline{\phi}_l(0)J(1)\phi_k(\infty)\rangle \equiv D_{\bar{l}J,k} = -qg_{k\bar{l}} \tag{F.12}$$

where q is the charge of ϕ_k. We also need the following value for the ψ integral for $h_\rho = s_\rho = 1$

$$\lim_{r\to 1}\int_{-\delta}^{+\delta} d\psi \frac{1}{(1-re^{2i\psi})}\left(-r^2 - \frac{e^{2i\psi}}{r}\right) = -\pi \tag{F.13}$$

So the contribution from the currents is equal to

$$-\frac{3}{c}(q+\overline{q})g_{i\bar{j}}g_{k\bar{l}} \tag{F.14}$$

All in all we get the following answer

$$C = g_{i\bar{j}}g_{k\bar{l}}\left(1 - \frac{3}{c}(q+\overline{q})\right) \tag{F.15}$$

G Current/Marginal operator OPE

Let us consider a chiral primary ϕ with $h = j^3 = 1/2$ and $\overline{h} = \overline{j}^3 = 1/2$. The marginal operator $\mathcal{O}(x)$ is the descendant of the chiral primary $\mathcal{O}(x) = G^{-k}_{-1/2}\overline{G}^{-l}_{-1/2}\cdot\phi(x)$. We want to compute the OPE of a current with the marginal operator. In general it will be

$$J^i(z)\mathcal{O}(w) = \sum_m \frac{(J^i_m\mathcal{O})(w)}{(z-w)^{m+1}} \tag{G.1}$$

So to compute the OPE we need to compute $J^i_m|\mathcal{O}\rangle$. We have

$$J^i_m|\mathcal{O}\rangle = \overline{G}^{-l}_{-1/2}\left([J^i_m, G^{-k}_{-1/2}] + G^{-k}_{-1/2}J^i_m\right)|\phi\rangle \tag{G.2}$$

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From the $\mathcal{N}=4$ algebra we have the following commutator of the modes

$$[J^i_m, G^{ak}_r] = \frac{1}{2}\sigma^i_{ba} G^{bk}_{m+r} \tag{G.3}$$

For $m=0$ we have $J^i_0|\mathcal{O}\rangle = 0$ since we already knew that $|\mathcal{O}\rangle$ is uncharged under the current algebra. For $m>0$ the second term in (G.2) is zero because $J^i_m|\phi\rangle = 0$, $m>0$. Also, from the commutators above we notice that the first term is proportional to a certain linear combination of

$$G^{cd}_{m-1/2} \tag{G.4}$$

If $m>0$ all of these operators annihilate the sate $|\phi\rangle$ because it is a primary, so finally we have

$$J^i_m|\mathcal{O}\rangle = 0, \qquad m \geq 0 \tag{G.5}$$

This proves that the OPE between the currents $J^i(z)$ and a marginal operator in $\mathcal{N}=(4,4)$ is completely regular.

There is in fact an alternative way to show this which does not rely on supersymmetry. Consider an exactly marginal operator in any theory which contains a non-abelian current algebra (which is preserved by the exactly marginal operator). The only singular terms in the OPE of a current with $\mathcal{O}$ arise from $J^i_0|\mathcal{O}\rangle$ and $J^i_{+1}|\mathcal{O}\rangle$. The first of these clearly vanishes, since $\mathcal{O}$ cannot be charged under the non-abelian current algebra. The second of these yields an operator of conformal weight $(0,1)$ which necessarily is an anti-holomorphic current. These cannot carry any charge under the holomorphic current algebra, whereas $J^i_{+1}|\mathcal{O}\rangle$ clearly does, and therefore $J^i_{+1}|\mathcal{O}\rangle = 0$ and the OPE between J^i and $\mathcal{O}$ has to be regular. Notice that this argument is completely general but fails for abelian current algebras.

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